Belgian Asteroseismology Group

STELLAR STABILITY AND ASTEROSEISMOLOGY

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Chapter 1

Introduction

It is usual, when studying a physical system, to first look at its equilibrium configurations. This approach is obviously justified in the case of stars. Stellar evolution studies teach us that during the major part of their lives stars are equilibrium structures which evolve very slowly under the effects of the changes of their chemical composition. It is then important to find out whether those equilibrium structures are stable or not. The study of stellar stability is therefore obviously complementary to the study of stellar structure.

When the equilibrium structure is unstable, we will study the development of the instability: time needed for its growth, and how it will appear. Very often this instability will appear as some kind of variability occuring on a timescale much smaller than the characteristic evolution timescale. The fraction of variable stars is small, but they are very much studied. The role played by RR Lyr stars and Cepeids to estimate astronomical distances is well known. In addition, the study of stellar oscillations provides information about internal stellar structure features which are not directly observable and allows therefore a test of the theory of stellar evolution.

The increase in the precision and in the time resolution of the observational instruments allows the observation of an increasing number of oscillations. Thousands of modes with periods close to 5 minutes (frequencies close to 3 mHz) have been identified on the Sun and their frequencies measured with an accuracy of the order of one μHz. The study of these data to determine the internal structure of the Sun is called helioseismology. The same technique applied to other stars is called asteroseismology.

Here we will only study gaseous stars. We will exclude the cases where the star or parts of it are in a state comparable to a solid state (white dwarfs under certain conditions, neutron stars). We will place ourselves in the context of non relativistic mechanics and the newtonian theory of gravitation, excluding the cases where it is justified to use a relativistic treatment (very condensed white dwarfs, neutron stars, black holes, supermassive stars).

It is well known that a mechanical system is stable if its equilibrium configuration corresponds to a minimum of the potential. This property can be extended to the case of a stellar configuration as long as only adiabatic transformations are considered. The extention of the method is not straightforward if non adiabatic terms are included. This method, based on the energy, has been rarely used and will not be presented in these

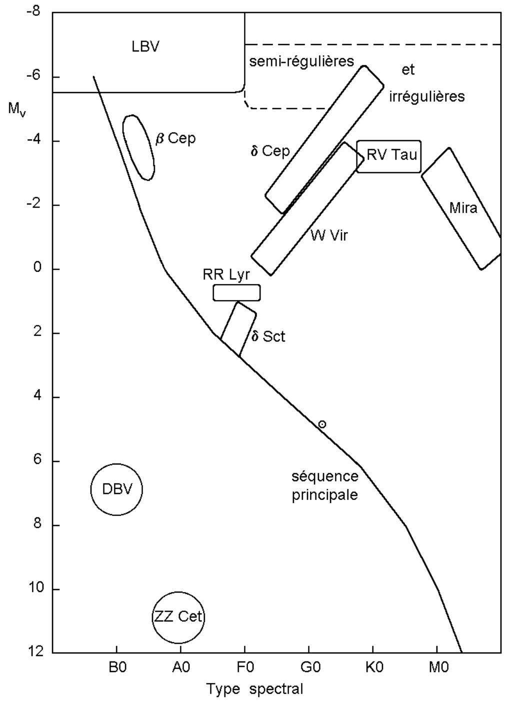

Figure 1.1: HR diagram with some types of variable stars.

lectures. In fact, a large fraction of these lectures will be dedicated to the small perturbations method. This one is easy to use, the properties of the solutions are quite well understood, and it is sufficient to explain most of the stellar variability linked to stellar stability. However, it is necessary to use a nonlinear theory to explain some behaviors (oscillations amplitudes, chaos, . . .).

We have approximately placed in an HR diagram some types of variable stars which will be refered to in these lectures (figure 1.1). We refer the reader to a textbook on variable stars for a detailed description of these different types of stars. We stress here that the δ Sct, RR lyr, δ Cep and W Vir variables lie in a region of the HR diagram called the *instability strip.*

References

Papers by Ledoux and Walraven (1958), Ledoux (1969), Cox (1974), Cox (1980), Unno et al. (1989) are classic references on this topic. A recent and short survey of the theory can be found in the papers by Gautschy and Saio (1995 and 1996). The energy-based method is descibed in a paper by Ledoux (1958).

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Chapter 2

Characteristic timescales

The different physical mechanisms operating in a star have extremely different timescales. It is important to know these timescales in order to model these mechanisms and use adequate approximations, or to choose correctly the integration timesteps in a numerical approach. We will give rough estimates for these timescales.

2.1 The dynamical timescale

It is well known that the hydrostatic equilibrium of a star results from the competition between gravity and pressure gradient forces. If the equilibrium between those two forces was broken, the star would respond on a timescale called the dynamical timescale. To estimate this timescale, assume that the pressure forces suddenly disappear and cease to oppose the gravity forces. Consider a unit mass element of matter. It would be subjected to the gravity force only and would undergo a freefall. Its motion would obey the equation

$$\ddot{r} = -\frac{Gm}{r^2}.$$

Let τ_{ff} be the characteristic timescale of this freefall. Using very rough estimates in the last equation, we have

$$\frac{R}{\tau_{ff}^2} \approx \frac{GM}{R^2},$$

which gives

$$\tau_{ff} \approx \sqrt{R^3/GM}.$$

It is also possible to estimate the timescale of the star reaction in response to the loss of hydrostatic equilibrium by assuming now that the gravity forces suddenly cease to exist. The pressure forces by themselves would dislocate the star. An element of matter would be accelerated outwards following the equation

$$\ddot{r} = -\frac{1}{\rho}\frac{dP}{dr}.$$

Let τ_{expl} be a characteristic timescale of this explosion. We can get a rough estimate of it by writing

$$\frac{R}{\tau_{expl}^2} \approx \frac{P}{\rho R} \approx \frac{c^2}{R}$$

where $c = \sqrt{\Gamma_1 P/\rho}$ is a characteristic sound speed in the star. We get

$$\tau_{expl} \approx \frac{R}{c}.$$

Because of the hydrostatic equilibrium, these two timescales are of the same order and define the dynamical timescale τ_{dyn}

$$\tau_{dyn} \approx \sqrt{\frac{R^3}{GM}} \approx \frac{1}{\sqrt{G\rho}}$$

where ρ is the mean stellar density. By comparing these two expressions we get an estimate of the sound speed in the star

$$c \approx \sqrt{\frac{GM}{R}}.$$

The table below gives a few typical values of the dynamical timescales in some types of stars.

Star	ρ ($\mathrm{g\,cm^{-3}}$)	$\tau_{dyn} = 1/\sqrt{G\rho}$
neutron stars	10^{15}	0.12 ms
white dwarfs	10^{6}	3.9 s
Sun	1.41	54 min
red supergiant	10^{-9}	3.9 yrs

Exercise

Show that the dynamical timescale also characterizes the circling motion of a satellite in a low orbit, as well as the rapid rotation of a star at the limit of disruption due to centrifugal forces.

2.2 Pulsation timescale

It is necessary to be careful when defining a characteristic pulsation timescale. We will see that a star has an infinity of pulsation modes, at all timescales. This should not be surprising: a simple vibrating string has a fundamental mode and an infinity of harmonics, whose periods tend to zero. For the most characteristic variables (cepheids, RR Lyr) the observed pulsation is in fact a radial acoustic mode of low order (fundamental or first harmonic). The estimate we will establish concerns this type of pulsation. Since it is a pressure mode (acoustic wave) we get an estimate for the pulsation characteristic timescale

$$\tau_{puls} \approx R/c.$$

The characteristic timescale obtained like that is again the dynamical timescale

$$\tau_{puls} \approx \tau_{dyn} \approx 1/\sqrt{G\rho}.$$

The product $\mathit{Period} \times \sqrt{G\rho}$ is therefore a dimensionless number of order unity, relatively independent of the stellar model. For the Sun (period of the fundamental mode: 63 minutes), it is 1.6. It is common to use the pulsation constant Q which is proportional to it and defined by

$$Q = \mathit{Period} \times \sqrt{\frac{\rho}{\rho_\odot}}.$$

Q has the dimensions of a time and detailed calculations show that as a rule we have

$$0.03 \text{ days} \leq Q \leq 0.08 \text{ days}.$$

In some unusual circumstances the estimate of the fundamental mode made above can be completely wrong (this is the case for a model close to dynamical instability).

2.3 The Kelvin-Helmholtz timescale

When a star slowly burns a nuclear fuel, the thermal equilibrium is the result of the competition between the rate of energy production and the rate of radiative energy loss. We get an estimate of the characteristic timescale of the thermal processes in the star by assuming that it suddenly loses its source of nuclear energy. It should then take the energy it radiates from its total energy E. The characteristic timescale of this process is called the Kelvin-Helmholz timescale. It is given by

$$\tau_{KH} \approx |E|/L.$$

In usual stellar conditions, the virial theorem provides a relationship between the total energy of the star and its potential energy Ω:

$$E = \frac{1}{2}\Omega.$$

On the other hand Ω is given by

$$\Omega = -q\frac{GM^2}{R}$$

where q is a factor close to unity, as shown in the following table

Model	q
homegeneous model	0.6
polytrope of index n	$3/(5-n)$
main sequence model	1.5

Thus we can write

$$\tau_{KH} \approx \frac{GM^2}{LR}.$$

For the Sun, this expression gives $3.1 \times 10^7\ yrs$.

It is interesting to compare the dynamical timescale to the thermal timescale

$$\frac{\tau_{dyn}}{\tau_{KH}} \approx \frac{LR^{5/2}}{G^{3/2}M^{5/2}}.$$

This ratio is much smaller than unity. For the Sun it is equal to 1.6×10^{-12}.

The comparison of these two timescales show that the thermal processes are very slow compared to the dynamical processes. In first approximation, it is therefore justified to ignore them when studying dynamical processes. These are global estimates, however, and locally thermal processes can have much smaller characteristic timescales, comparable to the dynamical timescale (e.g. in the external layers of the star).

2.4 The nuclear timescale

Consider a star on the main sequence, centrally burning its hydrogen, and try to estimate the time needed to significantly change its chemical composition. One gram of hydrogen liberates an energy of the order of $0.007c^2 \approx 6 \times 10^{18}$ ergs. If about 10 % of the stellar mass undergoes fusion, we get a nuclear lifetime of

$$\tau_{nuc} \approx 6 \times 10^{17} \frac{M}{L} \qquad \text{(CGS)}.$$

For the Sun, this gives 9.8×10^9 years. The ratio of the Kelvin-Helmholtz timescale to the characteristic nuclear timescale is given by

$$\frac{\tau_{KH}}{\tau_{nuc}} \approx 1.11 \times 10^{-25} \frac{M}{R} \qquad \text{(CGS)}.$$

This ratio is usually small. For the Sun it is equal to 3.2×10^{-3}. For a main sequence star, it is therefore justified to neglect the variations in the chemical composition when studying thermal processes.

Chapter 3

General equations

The study of stellar stability relies on the same physics as the construction of equilibrium models: hydrodynamics, radiative transport theory, thermodynamics, nuclear reactions theory, etc.

Two methods are commonly used to describe the motion of a fluid: the lagrangian and the eulerian methods. In the lagrangian method, each fluid particle is assigned a label and followed in its motion as in classical mechanics. The particle label could be for example its initial position $\vec{r}_0$. We will more generally use some vector $\vec{a}$ as label. The fluid is then described by the functions $\vec{r}(\vec{a},t),\ \rho(\vec{a},t),\ P(\vec{a},t),\ \ldots$.

In the eulerian description, particles are not followed one by one. Rather, at each position $\vec{r}$ the fluid is described by the functions $\vec{v}(\vec{r},t),\ \rho(\vec{r},t),\ P(\vec{r},t),\ \ldots$

The risk of confusion between the two descriptions comes from the fact that the same symbol, for example ρ, is used for two different functions, $\rho(\vec{a},t)$ and $\rho(\vec{r},t)$.

In particular, the time derivatives in these two formalisms do not have the same meaning: in the lagrangian formalism $\partial/\partial t$ is the time derivative following the motion of the fluid, while in the eulerian formalism this symbol represents the time derivative at a given point. Therefore we have

$$\begin{aligned} \frac{\partial \vec{r}}{\partial t} &= \vec{v} \qquad \text{in the Lagrange formalism,} \\ \frac{\partial \vec{r}}{\partial t} &= 0 \qquad \text{in the Euler formalism.} \end{aligned}$$

In the eulerian formalism we introduce a differential operator called the Stokes derivative, or derivative with respect to t following the fluid: D/Dt or d/dt. It is defined as follows:

$$\frac{dX}{dt} = \frac{\partial X}{\partial t} + \vec{v}\cdot \operatorname{grad} X.$$

It is clear that

$$\left(\frac{\partial X}{\partial t}\right)_{Lagrange} = \frac{dX}{dt}.$$

In practice, the two formalisms are often used simultaneously. To prevent any confusion between $(\partial/\partial t)_{Euler}$ and $(\partial/\partial t)_{Lagrange}$ we only use the notation $\partial/\partial t$ for $(\partial/\partial t)_{Euler}$ and we use d/dt for the operator $(\partial/\partial t)_{Lagrange}$. This convention is in agreement with the relationship between the Stokes derivative and the lagrangian time derivative, as seen above.

3.1 Continuity equation

The continuity equation represents the mass conservation. It can be written as

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \vec{v}) = 0 \qquad \text{or} \qquad \frac{d\rho}{dt} + \rho\, \mathrm{div}\, \vec{v} = 0.$$

In the lagrangian formalism it can also be written as

$$\rho X = \rho_0 X_0,$$

where X is the jacobian determinant

$$X = \left| \frac{\partial(x)}{\partial(a)} \right|.$$

We easily recover the first form of the equation if we note that

$$\frac{dX}{dt} = X\, \mathrm{div}\, \vec{v}.$$

3.2 Momentum equation

The momentum equation is given by

$$\begin{aligned} \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \mathrm{grad})\vec{v} &= -\,\mathrm{grad}\,\Phi - \frac{1}{\rho}\,\mathrm{grad}\,P \\ \text{or} \qquad \frac{d\vec{v}}{dt} &= -\,\mathrm{grad}\,\Phi - \frac{1}{\rho}\,\mathrm{grad}\,P. \end{aligned}$$

We did not include viscosity terms. Indeed in a star the effects of molecular viscosity are totally negligible and the fluid can be considered as perfect. In the convective zones, the velocity in the momentum equation is the bulk motion velocity, obtained by averaging over regions larger than the convective cells. The presence of convective motions is described by additional pressure and turbulent viscosity terms in the momentum equation. Contrary to molecular viscosity terms, they are usually not negligible. Unfortunately there is no good theory of non stationary convection. In this course, we will not consider the problem of non stationary convection. We will use the equations for the radiative zones.

3.3 Poisson equation

The gravitational potential field Φ obeys the Poisson equation

$$\Delta\Phi = 4\pi G\rho.$$

The physical solution is regular at infinity and is given by

$$\Phi(P) = -G\int \frac{\rho_Q\, dV_Q}{|PQ|}.$$

3.4 Energy conservation equation

The energy conservation equation is written as

$$\begin{aligned} T\left(\frac{\partial S}{\partial t} + \vec{v}\cdot \mathrm{grad}\, S\right) &= \epsilon - \frac{1}{\rho}\,\mathrm{div}\,\vec{F} \\ \text{or} \qquad T\frac{dS}{dt} &= \epsilon - \frac{1}{\rho}\,\mathrm{div}\,\vec{F}, \end{aligned}$$

where S is the entropy per unit mass, ϵ is the rate of nuclear energy generation per unit mass and $\vec{F}$ is the energy flux density. Remember that the amount of energy going through a surface dS per unit time is given by $\vec{F}\cdot\overrightarrow{dS}$ where $\overrightarrow{dS} = \vec{n}dS$, $\vec{n}$ being the unit vector normal to dS. In general $\vec{F}$ is the sum of a radiative term and a convective term $\vec{F} = \vec{F}_R + \vec{F}_C$.

3.5 Transport equation

In stellar interiors, the photon mean free path is extremely short compared to other characteristic length scales. The diffusion equation is therefore an excellent approximation of the radiative transport equation,

$$\vec{F}_R = -\lambda\,\mathrm{grad}\,T \qquad \text{with} \qquad \lambda = \frac{4acT^3}{3\kappa\rho}.$$

We note that the conduction flux in white dwarfs is given by a similar equation (with a different conduction coefficient λ). We will not give here the expression for the convective flux, for reasons given above.

3.6 Boundary conditions

Adequate boundary conditions must be imposed at the stellar surface. To zeroth order we can impose zero pressure and zero temperature at the surface of the star. To the next order, we consider that the photosphere (optical depth $\tau = 2/3$) defines the limit between the star interior and its atmosphere. We impose continuity at this limit. This separation between interior modelling and atmosphere modelling is useful because they deal differently with the transport equation.

3.7 Material equations

In addition to the partial differential equations and the associated boundary conditions described above, we also need the equations describing the behavior of the matter as a function of its chemical composition and the thermodynamic variables. These equations are sometimes called constitutive equations or material equations. They include the equation of state, the opacity κ and the nuclear energy generation rate ϵ. We will use the symbol χ to describe the chemical composition, and we will use ρ and T as independent thermodynamic variables. The properties of the matter can be decribed by relations of the form

$$P = P(\rho, T, \chi), \quad U = U(\rho, T, \chi), \quad \kappa = \kappa(\rho, T, \chi), \quad \epsilon = \epsilon(\rho, T, \chi).$$

Note that the nuclear energy generation rate is usually the result of many nuclear reactions whose rates depend on elements with very small abundances, very short lifetimes (on stellar evolution timescales) and not described by χ. It is only when these elements reach their equilibrium abundance values that ϵ can be considered a function of ρ, T et χ.

Chapter 4

Equilibrium configuration

A spherically symmetric configuration obeys the following equations, obtained from the general equations.

$$\frac{1}{\rho}\frac{dP}{dr} + \frac{d\Phi}{dr} = 0,$$
$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G\rho,$$
$$\epsilon - \frac{1}{\rho r^2}\frac{d}{dr}\left(r^2 F\right) = 0,$$
$$F = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr} \qquad \text{(radiative zone).}$$

The condition for stability of a radiative zone against convection is given by the Schwarzschild criterion

$$A \equiv \frac{d\ln\rho}{dr} - \frac{1}{\Gamma_1}\frac{d\ln P}{dr} < 0.$$

Let $m(r)$ be the mass of the sphere of radius r and $L(r)$ its luminosity.

$$m = \int_0^r 4\pi r^2 \rho\, dr\,,$$
$$L = 4\pi r^2 F\,.$$

The above equations can then be written under their usual form:

$$\begin{aligned}
\frac{dP}{dr} &= -\frac{Gm\rho}{r^2}\,, \\
\frac{dm}{dr} &= 4\pi r^2 \rho\,, \\
\frac{dL}{dr} &= 4\pi r^2 \rho\epsilon\,, \\
\frac{dT}{dr} &= -\frac{3\kappa\rho L}{16\pi r^2 acT^3} \qquad \text{(radiative zone).}
\end{aligned}$$

We note that in some evolution phases, the models are not in thermal equilibrium and there is a term involving the derivative of the entropy in the energy conservation equation. In what follows we will not consider out-of-thermal-equilibrium models.

We must add the boundary conditions to these equations. We must impose *natural* boundary conditions at the surface of the star and *artificial* boundary conditions at the center (because the singularity comes from the spherical coordinates themselves). At the center, m and L must be zero. At the surface (photosphere) the boundary conditions come from a detailed model of the outer layers. If the atmosphere is decribed by a temperature law of the form

$$T^4(\tau) = \frac{3}{4} T_e^4 \left(\tau + \frac{2}{3} \right),$$

we can adopt the following boundary conditions

$$\begin{aligned} &P = \frac{2}{3} \frac{GM}{\bar{\kappa} r^2}, \\ &T = T_e \qquad \text{or} \qquad L = 4\pi r^2 \sigma T^4 . \end{aligned}$$

References

For more details on equilibrium configurations, we refer to a course on Stellar Evolution, or the book by Kippenhahn and Weigert (1990).

Kippenhahn R., Weigert A., 1990. Stellar structure and evolution. Springer.

Chapter 5

Small perturbations method

We cannot solve exactly the equations describing a star. However, if the hydrodynamical variables remain close to their equilibrium values (or close to a known solution), it is possible to write any variable X as the sum of its equilibrium or unperturbed value X_0, and a small perturbation δX, so that

$$X = X_0 + \delta X.$$

Substituting these expressions into the hydrodynamical equations and keeping only terms up to the first order in the perturbed quantities, we get linear equations. These equations are much easier to study than the original equations, and their solutions are very useful approximations.

The small perturbations method makes it possible to study the stability of the stellar models against sufficiently small perturbations. It will not, however, give us any information on the stability against finite amplitude perturbations, on metastable states, or on limited cycles close to nonperturbed solutions. It will also not give us the pulsation amplitude of a variable star.

If the unperturbed configuration is stationary (as will be considered throughout this course), the coefficients of the linearized equations are time-independent. We can then write the general solution as a linear combination of simple solutions which depend exponentially on the time as e^{st} (s can be complex) and which are called normal modes. In the case of a mechanical system with a finite number of degrees of freedom, there is also a finite number of normal modes. Here, we have an infinite number of degrees of freedom and there exists an infinite number of normal modes of oscillation (as in the case of a vibrating string). A given mode is stable if $\Re s < 0$, unstable if $\Re s > 0$. A stellar configuration is stable if *all* its normal modes are stable, but it is unstable as soon as *one* mode is unstable. In the exceptional case where the stability of one or several modes is marginal ($\Re s = 0$), the other modes being stable, the linear analysis does not give the information on whether the considered model is stable or not.

Exercise

How do you write the perturbations of a sum, a product, a division ?

5.1 Lagrangian and eulerian perturbations

To the two descriptions of hydrodynamics correspond two types of perturbations. The lagrangian perturbation δX of X is described by

$$X(\vec{a},t) = X_0(\vec{a},t) + \delta X(\vec{a},t)$$

and the eulerian perturbation X' by

$$X(\vec{r},t) = X_0(\vec{r},t) + X'(\vec{r},t).$$

The relation between the lagrangian and the eulerian perturbations of a variable X is given by

$$\delta X = X' + \overrightarrow{\delta r} \cdot \mathrm{grad}\, X_0.$$

Since the perturbed variable X does not enter into the perturbation equations, but only the quantities X_0, δX and X', it is convenient to omit the subscript $_0$ in the unperturbed value of the variable. The above relation will then be written as

$$\delta X = X' + \overrightarrow{\delta r} \cdot \mathrm{grad}\, X.$$

Let x_i be some system of coordinates. The following relations are straightforward:

$$\begin{aligned} \frac{\partial X'}{\partial t} &= \left(\frac{\partial X}{\partial t}\right)', \\ \frac{\partial X'}{\partial x_i} &= \left(\frac{\partial X}{\partial x_i}\right)', \\ \frac{d\,\delta X}{dt} &= \delta\,\frac{dX}{dt}\,. \end{aligned}$$

It is important to be very careful when using lagrangian perturbations in the eulerian formalism (or vice-versa). Indeed, the relation between $\partial\,\delta X/\partial x_i$ and $\delta\,(\partial X/\partial x_i)$ is not straightforward. Using the relations given above, we have

$$\frac{\partial\,\delta X}{\partial x_i} = \delta\,\frac{\partial X}{\partial x_i} + \sum_j \frac{\partial\,\delta x_j}{\partial x_i}\frac{\partial X}{\partial x_j}.$$

In general it is easier to write the equations in the eulerian formalism. However, the lagrangian formalism is better to describe the radial oscillations.

Notes

The following remarks are often useful.

1) If the unperturbed configuration is static, ($\vec{v} = 0$ everywhere), then

$$\frac{d}{dt} = \frac{\partial}{\partial t}.$$

2) If the quantity X is independent of the coordinates in a given region of the unperturbed configuration, then we have

$$\delta X = X'$$

in that region.

5.2 Perturbation of the differential equations

Continuity equation:

$$\frac{\partial \rho'}{\partial t} + \mathrm{div}\left(\rho \frac{\partial\, \overrightarrow{\delta r}}{\partial t}\right) = 0 \qquad \text{or} \qquad \rho' + \mathrm{div}(\rho\, \overrightarrow{\delta r}) = 0.$$

Momentum equation:

$$\frac{\partial^2 \overrightarrow{\delta r}}{\partial t^2} = -\,\mathrm{grad}\,\Phi' + \frac{\rho'}{\rho^2}\,\mathrm{grad}\,P - \frac{1}{\rho}\,\mathrm{grad}\,P'.$$

Poisson equation:

$$\Delta\Phi' = 4\pi G\rho'.$$

Energy conservation equation:

$$T\left(\frac{\partial S'}{\partial t} + \vec{v}\cdot\mathrm{grad}\,S\right) = \epsilon' + \frac{\rho'}{\rho^2}\,\mathrm{div}\,\vec{F} - \frac{1}{\rho}\,\mathrm{div}\,\vec{F}'.$$

Radiative transfer equation:

$$\vec{F}' = -\lambda'\,\mathrm{grad}\,T - \lambda\,\mathrm{grad}\,T'.$$

The boundary conditions will be described for each particular case considered below.

In non-stationary conditions, the convective flux cannot be calculated by simply perturbing the expression giving the flux in the stationary case. Indeed, the convective cells have a finite lifetime and given their inertia (mechanical and thermal) they do not instantaneously respond to the changing conditions. It is only in the case where the lifetime of the perturbation is much larger then the lifetime of the convective cells that the convection can be considered as responding instantaneously to the changes; the usual expression for the convective flux can then be used.

5.3 Perturbation of the material equations

During the pulsation the relations $P = P(\rho, T)$ and $\kappa = \kappa(\rho, T)$, valid in the stationary case, continue to be satisfied at each moment. Therefore we have

$$\begin{aligned}\frac{\delta P}{P} &= P_\rho \frac{\delta\rho}{\rho} + P_T \frac{\delta T}{T},\\ \frac{\delta\kappa}{\kappa} &= \kappa_\rho \frac{\delta\rho}{\rho} + \kappa_T \frac{\delta T}{T},\end{aligned}$$

where

$$P_\rho = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_T,$$

$$P_T = \left(\frac{\partial \ln P}{\partial \ln T}\right)_\rho,$$
$$\kappa_\rho = \left(\frac{\partial \ln \kappa}{\partial \ln \rho}\right)_T,$$
$$\kappa_T = \left(\frac{\partial \ln \kappa}{\partial \ln T}\right)_\rho.$$

These quantities can be calculated or deduced from the tables used for the calculation of the stellar models. The coefficients P_ρ and P_T can be calculated as functions of the coefficients Γ , which we will often use.

$$\Gamma_1 = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_S, \qquad \frac{\Gamma_2 - 1}{\Gamma_2} = \left(\frac{\partial \ln T}{\partial \ln P}\right)_S, \qquad \Gamma_3 - 1 = \left(\frac{\partial \ln T}{\partial \ln \rho}\right)_S.$$

Only two of these three coefficients are independent. Indeed, it is easy to show that

$$\frac{\Gamma_2 - 1}{\Gamma_2} = \frac{\Gamma_3 - 1}{\Gamma_1}.$$

Let U be the internal energy per unit mass. We know that

$$\delta U = T\delta S - P\delta V,$$

i.e.,

$$T = \left(\frac{\partial U}{\partial S}\right)_V \quad \text{and} \quad P = -\left(\frac{\partial U}{\partial V}\right)_S.$$

Using

$$\frac{\partial^2 U}{\partial S \partial V} = \frac{\partial^2 U}{\partial V \partial S},$$

we have

$$-\left(\frac{\partial P}{\partial S}\right)_V = \left(\frac{\partial T}{\partial V}\right)_S.$$

Using ρ and S as independent variables, this last relation can be rewritten as

$$\left(\frac{\partial \ln P}{\partial S}\right)_\rho = \frac{(\Gamma_3 - 1)\rho T}{P}.$$

We can of course write

$$\frac{\delta P}{P} = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_S \frac{\delta\rho}{\rho} + \left(\frac{\partial \ln P}{\partial S}\right)_\rho \delta S,$$
$$\frac{\delta T}{T} = \left(\frac{\partial \ln T}{\partial \ln \rho}\right)_S \frac{\delta\rho}{\rho} + \left(\frac{\partial \ln T}{\partial S}\right)_\rho \delta S.$$

The coefficient $(\partial \ln T/\partial S)_\rho$ is in fact the inverse of c_v, the specific heat at constant volume per unit mass. The above relations can therefore be written as

$$\frac{\delta P}{P} = \Gamma_1 \frac{\delta\rho}{\rho} + \frac{(\Gamma_3 - 1)c_v\rho T}{P}\frac{\delta S}{c_v},$$
$$\frac{\delta T}{T} = (\Gamma_3 - 1)\frac{\delta\rho}{\rho} + \frac{\delta S}{c_v}.$$

Eliminating δS we get the linearized equation of state

$$\frac{\delta P}{P} = \left[\Gamma_1 - \frac{(\Gamma_3 - 1)^2 c_v \rho T}{P}\right] \frac{\delta\rho}{\rho} + \frac{(\Gamma_3 - 1) c_v \rho T}{P} \frac{\delta T}{T},$$

i.e.,

$$\begin{aligned} P_\rho &= \Gamma_1 - \frac{(\Gamma_3 - 1)^2 c_v \rho T}{P}, \\ P_T &= \frac{(\Gamma_3 - 1) c_v \rho T}{P}. \end{aligned}$$

The case of ϵ, the rate of nuclear energy generation, is completely different. If the pulsation time is long compared to the reactants lifetimes, the abundances or the different nuclei participating to the nuclear reactions assume their instantaneous equilibrium values and we can write

$$\frac{\delta\epsilon}{\epsilon} = \epsilon_\rho \frac{\delta\rho}{\rho} + \epsilon_T \frac{\delta T}{T},$$

with

$$\epsilon_\rho = \left(\frac{\partial \ln \epsilon}{\partial \ln \rho}\right)_T \quad \text{and} \quad \epsilon_T = \left(\frac{\partial \ln \epsilon}{\ln T}\right)_\rho .$$

However, in general, this will not be the case and it will be necessary to take into account the details of the nuclear reactions and to study the perturbed solutions to the equations describing their kinetics. We then get

$$\frac{\delta\epsilon}{\epsilon} = \epsilon_\rho(\sigma) \frac{\delta\rho}{\rho} + \epsilon_T(\sigma) \frac{\delta T}{T},$$

where the coefficients ϵ_ρ and ϵ_T are functions of the pulsation frequency σ. These coefficients are usually complex numbers (phase shift). They are often designated by μ_{eff} and ν_{eff} in the literature.

Exercises

1. Derive the thermodynamic coefficients of a perfect gas, of a mixture of gas and radiation, of partially ionized gas.

2. Derive the coefficients $\epsilon_\rho(\sigma)$ and $\epsilon_T(\sigma)$ for the proton-proton chain and for the carbon cycle.

Chapter 6

Adiabatic perturbations

We saw that the characteristic timescale for energy transfer in a star is much larger than the characteristic dynamical timescale (except in the external layers of the star). It is therefore natural, as a first approximation, to neglect transport phenomena and energy production for perturbations which evolve on a dynamical timescale. The energy conservation equation then becomes

$$\delta S = 0.$$

This equation describes an adiabatic perturbation. As a consequence, there is a simple relation between the variations of density and pressure:

$$\frac{\delta P}{P} = \Gamma_1 \frac{\delta \rho}{\rho} \qquad \text{or} \qquad \delta P = c^2 \delta \rho,$$

where $c = \sqrt{\Gamma_1 P/\rho}$ is the sound speed.

Of course, in the adiabatic approximation, the equation of energy transport is no longer needed. We will then keep the adiabatic equation, the continuity equation, the momentum equation, and the Poisson equation. We will show that this simplified problem can be written as a self-adjoint problem.

All the perturbed quantities can be easily written as functions of the displacement $\overrightarrow{\delta r}$. The continuity equation can be written as

$$\rho' + \operatorname{div}(\rho \overrightarrow{\delta r}) = 0 \quad \text{or} \quad \delta\rho + \rho \operatorname{div} \overrightarrow{\delta r} = 0$$

which gives

$$\rho' = -\operatorname{div}(\rho \overrightarrow{\delta r}) \quad \text{or} \quad \delta\rho = -\rho \operatorname{div} \overrightarrow{\delta r}.$$

The adiabatic equation then gives δP or P'

$$\delta P = -\Gamma_1 P \operatorname{div} \overrightarrow{\delta r} \quad \text{or} \quad P' = -\Gamma_1 P \operatorname{div} \overrightarrow{\delta r} - \overrightarrow{\delta r} \cdot \operatorname{grad} P.$$

Finally the Poisson equation

$$\Delta \Phi' = 4\pi G \rho'$$

has the following solution

$$\Phi'(P) = -G \int \frac{\rho'_Q \, dV_Q}{|PQ|} = G \int \frac{\operatorname{div}(\rho \overrightarrow{\delta r})_Q \, dV_Q}{|PQ|}.$$

The momentum equation

$$\frac{d^2 \overrightarrow{\delta r}}{dt^2} = -\operatorname{grad} \Phi' + \frac{\rho'}{\rho^2} \operatorname{grad} P - \frac{1}{\rho} \operatorname{grad} P'$$

can then be written as

$$\frac{d^2 \overrightarrow{\delta r}}{dt^2} = \mathcal{L} \overrightarrow{\delta r},$$

where $\mathcal{L}$ is the linear operator defined by

$$\begin{aligned} \mathcal{L}\overrightarrow{\delta r} &= -G \operatorname{grad}_P \int \frac{\operatorname{div}(\rho \overrightarrow{\delta r})_Q \, dV_Q}{|PQ|} - \frac{1}{\rho^2} \operatorname{div}(\rho \overrightarrow{\delta r}) \operatorname{grad} P \\ &\quad + \frac{1}{\rho} \operatorname{grad}(\Gamma_1 P \operatorname{div} \overrightarrow{\delta r}) + \frac{1}{\rho} \operatorname{grad}(\overrightarrow{\delta r} \cdot \operatorname{grad} P). \end{aligned}$$

Let us consider the functional space of vector fields continuously derivable and define the scalar product of two such vector fields by

$$(\vec{u}, \vec{v}) = \int_V \rho \vec{u} \cdot \bar{\vec{v}} \, dV = \overline{(\vec{v}, \vec{u})}.$$

In the integral, V stands for a volume which is large enough to include the whole stellar configuration. We will use S to designate the surface delimiting this volume.

We will show that $\mathcal{L}$ is a symmetrical operator. As it is obviously real, it will therefore be hermitian (i.e. self-adjoined). For two real fields $\vec{u}$ and $\vec{v}$, we will show that we have

$$(\mathcal{L}\vec{u}, \vec{v}) = (\vec{u}, \mathcal{L}\vec{v}).$$

To prove this property we will transform some integrals in the following way:

$$\begin{aligned} \int_V \vec{a} \cdot \operatorname{grad} \alpha \, dV &= \int_V [\operatorname{div}(\alpha \vec{a}) - \alpha \operatorname{div} \vec{a}] \, dV \\ &= \oint_S \alpha \vec{a} \cdot \overrightarrow{dS} - \int_V \alpha \operatorname{div} \vec{a} \, dV = -\int_V \alpha \operatorname{div} \vec{a} \, dV. \end{aligned}$$

This last equality is justified only if the surface integral is zero, which must be checked for each case.

$$\begin{aligned} (\mathcal{L}\vec{u}, \vec{v}) &= -G \int dV_P \, \rho \vec{v} \cdot \operatorname{grad}_P \int \frac{\operatorname{div}(\rho \vec{u})_Q}{|PQ|} \, dV_Q - \int \frac{1}{\rho} (\vec{v} \cdot \operatorname{grad} P) \operatorname{div}(\rho \vec{u}) \, dV \\ &\quad + \int \vec{v} \cdot \operatorname{grad}(\Gamma_1 P \operatorname{div} \vec{u}) \, dV + \int \vec{v} \cdot \operatorname{grad}(\vec{u} \cdot \operatorname{grad} P) \, dV \\ &= A + B + C + D. \end{aligned}$$

We have

$$
\begin{aligned}
A &= G \iint \frac{\mathrm{div}(\rho\vec{u})_Q \,\mathrm{div}(\rho\vec{v})_P}{|PQ|}\, dV_P\, dV_Q\,, \\
B &= -\int (\mathrm{div}\,\vec{u})\vec{v}\cdot \mathrm{grad}\, P\, dV - \int \frac{1}{\rho}(\vec{u}\cdot \mathrm{grad}\,\rho)(\vec{v}\cdot \mathrm{grad}\, P)\, dV = B_1 + B_2\,, \\
C &= -\int \Gamma_1 P(\mathrm{div}\,\vec{u})(\mathrm{div}\,\vec{v})\, dV\,, \\
D &= -\int (\vec{u}\cdot \mathrm{grad}\, P)\,\mathrm{div}\,\vec{v}\, dV\,.
\end{aligned}
$$

The expressions A, B_2 and C are obviously symmetrical (note for B_2 that $\mathrm{grad}\,\rho \parallel \mathrm{grad}\, P$). The same can be said for $B_1 + D$.

Let us come back to the momentum equation. The interesting solutions for us must also satisfy the linear boundary conditions, which are homogeneous and time-independent. The problem therefore has simple solutions of the form $\vec{\xi}(\vec{r})e^{st}$. These solutions are the normal modes of oscillation of the star. The $\vec{\xi}(\vec{r})$ obey the following equation (from now on we will omit the arrow above ξ)

$$\mathcal{L}\xi = s^2\xi.$$

The function ξ is called the eigenfunction, and s the eigenvalue. In fact, it is really $\lambda = s^2$ which should be called the eigenvalue. The problem admits an infinite number of eigenvalues. The analogy with a mechanical system with a finite number of degrees of freedom is obvious. We will accept without proof that these eigenfunctions form a complete set, i.e. that they constitute a basis to write any perturbation.

The fact that $\mathcal{L}$ is hermitian has several interesting consequences.

1°) The eigenvalues λ are real. Indeed, if ξ is the eigenvalue associated to λ

$$\mathcal{L}\xi = \lambda\xi.$$

We take the product with ξ

$$(\mathcal{L}\xi, \xi) = \lambda(\xi, \xi).$$

But

$$(\mathcal{L}\xi, \xi) = (\xi, \mathcal{L}\xi) = \overline{(\mathcal{L}\xi, \xi)},$$

so that $\lambda = (\mathcal{L}\xi, \xi)/(\xi, \xi)$ is real. Therefore, the eigenfunctions are real up to a constant multiplying factor. We will assume hereafter that we have chosen real eigenfunctions. If $\lambda > 0$, $s = \pm\sqrt{\lambda}$ and one of the two modes is unstable. This instability is called a dynamical instability. If $\lambda < 0$, $s = \pm i\sqrt{-\lambda}$ and the linear analysis does not establish a dynamical instability. A model in which all the modes have $\lambda < 0$ is dynamically stable.

2°) The eigenfunctions associated to different eigenvalues are orthogonal. Indeed,

$$
\begin{gathered}
\mathcal{L}\xi_i = \lambda_i\xi_i \quad \text{and} \quad \mathcal{L}\xi_j = \lambda_j\xi_j\,, \\
\lambda_i(\xi_i, \xi_j) = (\mathcal{L}\xi_i, \xi_j) = (\xi_i, \mathcal{L}\xi_j) = \lambda_j(\xi_i, \xi_j)\,.
\end{gathered}
$$

If λ_i and λ_j are distinct, we have $(\xi_i, \xi_j) = 0$. On the other hand, a family of independent eigenfunctions associated to the same eigenvalue can always be orthogonalized. We can therefore consider that all the eigenfunctions are orthogonal to each other.

3°) The eigenvalues and eigenfunctions obey to a variational principle. Consider the functional

$$\Lambda(\xi) = \frac{(\mathcal{L}\xi, \xi)}{(\xi, \xi)}$$

and its first variation

$$\delta\Lambda = \frac{1}{(\xi, \xi)^2} \left\{ [(\mathcal{L}\delta\xi, \xi) + (\mathcal{L}\xi, \delta\xi)](\xi, \xi) - (\mathcal{L}\xi, \xi)[(\delta\xi, \xi) + (\xi, \delta\xi)] \right\} .$$

The condition $\delta\Lambda = 0$ is equivalent to

$$(\xi, \xi)(\mathcal{L}\xi, \delta\xi) - (\mathcal{L}\xi, \xi)(\xi, \delta\xi) + (\xi, \xi)(\mathcal{L}\overline{\xi}, \overline{\delta\xi}) - (\mathcal{L}\xi, \xi)(\overline{\xi}, \overline{\delta\xi}) = 0 .$$

Since $\delta\xi$ is arbitrary, we have

$$\mathcal{L}\xi = \frac{(\mathcal{L}\xi, \xi)\xi}{(\xi, \xi)} = \Lambda(\xi)\xi .$$

The functional $\Lambda(\xi)$ is therefore stationary when ξ is an eigenfunction, and its corresponding eigenvalue is $\Lambda(\xi)$. From this variational principle, it is possible to deduce a method to calculate the eigenmodes. It has been used, although not very frequently. It is also possible to deduce a method to improve the calculation of an eigenvalue: a gross approximation of ξ will give a good approximation $\Lambda(\xi)$ for the eigenvalue λ.

For a stable mode, we will write $s = -i\sigma$. Note that the average kinetic energy of a mode with an angular frequency σ is given by

$$\overline{T} = \frac{\sigma^2}{4}(\xi, \xi) .$$

References

The operator $\mathcal{L}$ is just a particular case of the operator used by Lynden-Bell and Ostriker (1967). We follow rather closely Cox (1980), chapter 5.

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Chapter 7

Radial oscillations

7.1 Differential equations

In the case of radial oscillations, it is particularly useful to use lagrangian perturbations because it is possible to replace the spatial coordinate r by a *lagrangian* coordinate m. The advantage of this change of coordinates is that the operator $\partial/\partial m$ commutes with the operator of lagrangian perturbation δ.

We have a relation between $\partial/\partial m$ and $\partial/\partial r$

$$\frac{\partial}{\partial m} = \frac{1}{4\pi r^2 \rho}\frac{\partial}{\partial r}.$$

We now write the general equations for a spherically symmetric fluid. We have already shown that the Poisson equation can be integrated once to give

$$\frac{\partial \Phi}{\partial r} = \frac{Gm}{r^2}.$$

The continuity, momentum, energy conservation, and radiative transfer equations are given by

$$\begin{aligned}
\frac{1}{\rho}\frac{d\rho}{dt} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v) &= 0 & \text{or} \quad & \frac{1}{\rho}\frac{d\rho}{dt} + 4\pi\rho\frac{\partial}{\partial m}(r^2 v) = 0,\\
\frac{dv}{dt} &= -\frac{Gm}{r^2} - \frac{1}{\rho}\frac{\partial P}{\partial r} & \text{or} \quad & \frac{dv}{dt} = -\frac{Gm}{r^2} - 4\pi r^2\frac{\partial P}{\partial m},\\
T\frac{dS}{dt} &= \epsilon - \frac{1}{\rho r^2}\frac{\partial}{\partial r}(r^2 F) & \text{or} \quad & T\frac{dS}{dt} = \epsilon - \frac{\partial L}{\partial m},\\
F &= -\frac{4acT^3}{3\kappa\rho}\frac{\partial T}{\partial r} & \text{or} \quad & L = -\frac{64\pi^2 r^4 acT^3}{3\kappa}\frac{\partial T}{\partial m}.
\end{aligned}$$

The lagrangian perturbation operator δ commutes with the two operators d/dt and $\partial/\partial m$ which appear in those equations. It is therefore easy to write the perturbed equations

$$\frac{\delta\rho}{\rho} + 4\pi\rho\frac{\partial}{\partial m}(r^2\delta r) = 0,$$

$$\frac{d^2\delta r}{dt^2} = 2\frac{Gm}{r^2}\frac{\delta r}{r} - 8\pi r^2 \frac{\partial P}{\partial m}\frac{\delta r}{r} - 4\pi r^2 \frac{\partial\,\delta P}{\partial m},$$
$$T\frac{d\,\delta S}{dt} = \delta\epsilon - \frac{\partial\,\delta L}{\partial m},$$
$$\delta L = -\frac{64\pi^2 r^4 acT^3}{3\kappa}\left\{\frac{\partial T}{\partial m}\left(4\frac{\delta r}{r} + 3\frac{\delta T}{T} - \frac{\delta\kappa}{\kappa}\right) + \frac{\partial\,\delta T}{\partial m}\right\}.$$

We now introduce again the operator $\partial/\partial r$ and we get

$$\begin{aligned}
\frac{\partial}{\partial r}\left(\frac{\delta r}{r}\right) &= -\frac{1}{r}\left(3\frac{\delta r}{r} + \frac{\delta\rho}{\rho}\right), \\
\frac{\partial}{\partial r}\left(\frac{\delta P}{P}\right) &= -\frac{1}{P}\frac{dP}{dr}\left\{\frac{\delta P}{P} + 4\frac{\delta r}{r} - \frac{r^3}{Gm}\frac{d^2}{dt^2}\left(\frac{\delta r}{r}\right)\right\}, \\
\frac{\partial}{\partial r}\left(\frac{\delta L}{L}\right) &= -\frac{1}{L}\frac{dL}{dr}\left(\frac{\delta L}{L} - \frac{\delta\epsilon}{\epsilon}\right) - \frac{4\pi r^2 \rho T}{L}\frac{d\,\delta S}{dt}, \\
\frac{\partial}{\partial r}\left(\frac{\delta T}{T}\right) &= -\frac{1}{T}\frac{dT}{dr}\left(4\frac{\delta r}{r} + 4\frac{\delta T}{T} - \frac{\delta L}{L} - \frac{\delta\kappa}{\kappa}\right).
\end{aligned}$$

These equations must be completed by the perturbed boundary conditions and material equations.

The coefficients of these linear equations are time-independent. We will therefore look for simple solutions of the form

$$\frac{\delta X}{X}(r,t) = \frac{\delta X}{X}(r)e^{st}.$$

Note that for simplicity we will continue to write $\delta X/X$ for the r-dependent function. The partial differential equations can then be reduced to ordinary differential equations depending on the parameter s.

$$\begin{aligned}
\frac{d}{dr}\left(\frac{\delta r}{r}\right) &= -\frac{1}{r}\left(3\frac{\delta r}{r} + \frac{\delta\rho}{\rho}\right), \\
\frac{d}{dr}\left(\frac{\delta P}{P}\right) &= -\frac{1}{P}\frac{dP}{dr}\left\{\frac{\delta P}{P} + \left(4 - \frac{r^3 s^2}{Gm}\right)\frac{\delta r}{r}\right\}, \\
\frac{d}{dr}\left(\frac{\delta L}{L}\right) &= -\frac{1}{L}\frac{dL}{dr}\left(\frac{\delta L}{L} - \frac{\delta\epsilon}{\epsilon}\right) - \frac{4\pi r^2 c_v \rho T}{L} s \frac{\delta S}{c_v}, \\
\frac{d}{dr}\left(\frac{\delta T}{T}\right) &= -\frac{1}{T}\frac{dT}{dr}\left(4\frac{\delta r}{r} + 4\frac{\delta T}{T} - \frac{\delta L}{L} - \frac{\delta\kappa}{\kappa}\right).
\end{aligned}$$

The choice of a coordinate system presenting a singularity at the star center introduces a regular singularity at this point. We will impose the condition that the physical quantities δr, δP, ... remain regular. A simple power series expansion gives the regularity conditions at the center:

$$\begin{aligned}
3\frac{\delta r}{r} + \frac{\delta\rho}{\rho} &= 0, \\
\epsilon\left(\frac{\delta L}{L} - \frac{\delta\epsilon}{\epsilon}\right) + sT\,\delta S &= 0.
\end{aligned}$$

At the surface, the simplest models (polytropes) impose vanishing pressure and temperature conditions. In this case, the system of differential equations presents a surface singularity. By requiring that the physical quantities remain regular at the surface we get the following boundary conditions

$$\begin{aligned} \frac{\delta P}{P} + \left(4 - \frac{R^3 s^2}{GM}\right)\frac{\delta r}{r} &= 0, \\ 4\frac{\delta r}{r} + 4\frac{\delta T}{T} - \frac{\delta L}{L} - \frac{\delta \kappa}{\kappa} &= 0. \end{aligned}$$

If the equilibrium model was joined to a model atmosphere, we would impose that the perturbations of the interior are continuously joined to the perturbations of the atmosphere. If the above mechanical condition is relatively satisfactory, the radiative condition can be easily improved. Assume that the thermal structure of the atmosphere can be described by the Eddington approximation at all times

$$T^4 = \frac{3}{4}T_e^4(\tau + \frac{2}{3}) \approx \frac{3L}{16\pi r^2\sigma}(\tau + \frac{2}{3}).$$

In addition

$$\tau = \int_r^\infty \kappa\rho\, dr \approx \frac{\kappa\,\Delta m}{4\pi r^2}.$$

We then get

$$4\frac{\delta T}{T} + 2\frac{\delta r}{r} - \frac{\delta L}{L} - \frac{\tau}{\tau + 2/3}\left(\frac{\delta\kappa}{\kappa} - 2\frac{\delta r}{r}\right) = 0.$$

During the pulsation, the photosphere level moves through the fluid. It is therefore incorrect to directly perturb the equation

$$L = 4\pi R^2 \sigma T_e^4$$

and to apply it to the level corresponding to $\tau = 2/3$ of the equilibrium model.

The search for radial oscillation modes comes down to solving an eigenvalue-boundary conditions system of fourth order differential equations.

7.2 Integral equations

We can derive integral equations from the above equations. These integral equations will prove very useful for the remainder of this course. Let us write the continuity, momentum, and energy equations as

$$\begin{aligned} \frac{d}{dr}\left(r^2\delta r\right) &= -r^2\frac{\delta\rho}{\rho}, \\ \frac{d\,\delta P}{dr} + \left(\rho s^2 + \frac{4}{r}\frac{dP}{dr}\right)\delta r &= 0, \\ s c_v T\frac{\delta S}{c_v} &= \delta\epsilon - \frac{d\,\delta L}{dm}. \end{aligned}$$

We take the product of the momentum equation by $4\pi r^2\,\overline{\delta r}$ and integrate it over the stellar radius

$$s^2\int 4\pi r^2\rho|\delta r|^2 dr + \int 4\pi r^2\,\overline{\delta r}\left(\frac{d\,\delta P}{dr} + \frac{4}{r}\frac{dP}{dr}\delta r\right) dr = 0.$$

Using the continuity equation and integrating by parts, this equation can be written as

$$s^2\int |\delta r|^2 dm + \int\left\{\frac{\delta P}{\rho}\frac{\overline{\delta\rho}}{\rho} + \frac{4r}{\rho}\frac{dP}{dr}\left|\frac{\delta r}{r}\right|^2\right\}dm = 0.$$

This equation will be used to study vibrational stability. Let's write δP as function of $\delta\rho$ and of δS, then write δS using the energy conservation equation.

$$s^2\int |\delta r|^2 dm + \int\left\{\frac{\Gamma_1 P}{\rho}\left|\frac{\delta\rho}{\rho}\right|^2 + \frac{4r}{\rho}\frac{dP}{dr}\left|\frac{\delta r}{r}\right|^2\right\}dm + \frac{1}{s}\int(\Gamma_3-1)\frac{\overline{\delta\rho}}{\rho}\left(\delta\epsilon - \frac{d\,\delta L}{dm}\right)dm = 0.$$

The s-independent term can be written in a different, interesting form. Its first term can be developed as follows

$$\begin{aligned}
\int\frac{\Gamma_1 P}{\rho}\left|\frac{\delta\rho}{\rho}\right|^2 dm &= \int\frac{4\pi\Gamma_1 P}{r^2}\left|\frac{d}{dr}(r^2\delta r)\right|^2 dr = \cdots \\
&= \int\left\{4\pi\Gamma_1 P r^4\left|\frac{d}{dr}\left(\frac{\delta r}{r}\right)\right|^2 - 12\pi r^3\left|\frac{\delta r}{r}\right|^2\frac{d}{dr}(\Gamma_1 P)\right\}dr \\
&= \int\left\{\frac{\Gamma_1 P r^2}{\rho}\left|\frac{d}{dr}\left(\frac{\delta r}{r}\right)\right|^2 - \frac{r}{\rho}\frac{d}{dr}(3\Gamma_1 P)\left|\frac{\delta r}{r}\right|^2\right\}dm.
\end{aligned}$$

The s-independent term can then be written as

$$\int\left\{\frac{\Gamma_1 P r^2}{\rho}\left|\frac{d}{dr}\left(\frac{\delta r}{r}\right)\right|^2 - \frac{r}{\rho}\frac{d}{dr}[(3\Gamma_1-4)P]\left|\frac{\delta r}{r}\right|^2\right\}dm.$$

We have obtained the cubic equation

$$s^3 + As + B = 0,$$

with

$$\begin{aligned}
A &= \int\left\{c^2\left|\frac{\delta\rho}{\rho}\right|^2 - 4\frac{Gm}{r^3}|\delta r|^2\right\}dm\Big/\int|\delta r|^2 dm \\
&= \int\left\{c^2 r^2\left|\frac{d}{dr}\left(\frac{\delta r}{r}\right)\right|^2 - \frac{r}{\rho}\frac{d}{dr}[(3\Gamma_1-4)P]\left|\frac{\delta r}{r}\right|^2\right\}dm\Big/\int|\delta r|^2 dm, \\
B &= \int(\Gamma_3-1)\frac{\overline{\delta\rho}}{\rho}\left(\delta\epsilon - \frac{d\,\delta L}{dm}\right)dm\Big/\int|\delta r|^2 dm.
\end{aligned}$$

7.3 Dynamical modes and secular modes

We notice that the eigenvalue appears twice in the differential equations coefficients. If we write these in their dimensionless form, the first one occurs in the momentum equation as $r^3 s^2/Gm$, and the second one in the energy conservation equation as $4\pi r^3 c_v \rho T s/L$.

To each of these is associated a family of modes: the dynamical modes and the secular modes. The characteristic times of these modes are given by the corresponding coefficients of s. For the dynamical modes the characteristic time is the dynamical time, and for the secular modes it is the Kelvin-Helmholtz time

$$\sqrt{\frac{r^3}{Gm}} \approx \tau_{dyn}, \qquad \frac{4\pi r^3 c_v \rho T}{L} \approx \tau_{KH}.$$

These statements can be justified by numerous calculations. We will try to make them plausible by evaluating the order of magnitude of the roots of the cubic equation we have established above.

We will assume that the eigenfunctions are of the same order of magnitude to get order-of-magnitude estimates for the coefficients A and B (this approximation relies on the results of numerous numerical integrations). We get

$$A \approx 1/\tau_{dyn}^2 \qquad \text{et} \qquad B \approx \frac{1}{\tau_{dyn}^2 \tau_{KH}}.$$

We will therefore write $A = A'/\tau_{dyn}^2$ and $B = B'/\tau_{dyn}^2 \tau_{KH}$, where A' and B' are dimensionless numbers of order unity. The cubic equation can then be written as

$$s'^3 + A's' + \alpha B' = 0,$$

where $s = s'/\tau_{dyn}$ and $\alpha = \tau_{dyn}/\tau_{HK} \ll 1$. It is easy to see that this equation has two roots of order unity, which can be approximated by setting $\alpha = 0$,

$$s' = \pm\sqrt{-A'}$$

and one root of order α. We get an approximation of the latter by neglecting the cubic term or by noticing that the product of the roots is equal to $-\alpha B'$,

$$s' = -\alpha B'/A'.$$

We therefore get two roots of order $1/\tau_{dyn}$ and one root of order $1/\tau_{KH}$ for s. The first two roots are associated to perturbations whose characteristic evolution time is the dynamical time τ_{dyn}. These modes are called dynamical modes. The third root is associated to perturbations evolving on a much longer time-scale, the Kelvin-Helmholtz time-scale. These modes are called secular modes. Their existence is linked to the B coefficient which contains all the non adiabatic effects.

The preceding argument has however a weakness. The coefficients of the cubic equation are calculated starting from unknown eigenfunctions which will of course be different for dynamical and for secular modes. But the existence of two types of radial modes can also be justified through other methods: local analysis of the system of differential equations, use of the simple model such as the one-layer model of Baker.

The distinction between the two families of modes can generally be made without any ambiguity. There are however cases where the modes cannot be attributed to one or the other of these families. This happens when the thermal timescale becomes comparable to the dynamical timescale (very luminous stars, models close to dynamical instability).

References

The surface radiative boundary condition used above was obtained Baker and Kippenhahn (1965).

We have used a diffusion equation to describe the radiative transport. In the external layers of the star, this approximation is sometimes unsatisfactory. It is then possible to use the Eddington approximation. More details on this topic can be found in Unno and Spiegel (1966) and in Christensen-Dalsgaard and Frandsen (1983).

The cubic equation in s was obtained by Ledoux (1963 and 1969).

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Chapter 8

Adiabatic radial oscillations

The periods of the dynamical modes of a relatively low order are of the order of the dynamical timescale and are much shorter than the characteristic timescale of thermal energy transport. This justifies, as a first order approximation, the use of the adiabatic approximation to study these modes. If this study shows an instability, the star is said to be *dynamically unstable*. In the opposite case, it is said to be *dynamically stable*. If the stability of the star is not marginal, the adiabatic approximation will give a very good approximation of the eigenvalues of the normal modes. The adiabatic eigenfunctions will also constitute a very good approximation, except in the exterior layers where the adiabatic approximation is no longer justified (the local thermal characteristic timescale is no longer large compared to the dynamical timescale).

As we have shown, the adiabatic approximation generally leads to a self-adjoint problem where the eigenvalues s^2 are real. In the case of dynamical stability, s is purely imaginary. The adiabatic analysis therefore cannot predict whether the corresponding mode is excited or damped. The answer to that question can only be found by considering the non adiabatic terms and constitutes the problem of the *vibrational stability*, which will be discussed in the next chapter.

In the adiabatic approximation, $\delta P/P$ and $\delta\rho/\rho$ are related through

$$\frac{\delta P}{P} = \Gamma_1 \frac{\delta\rho}{\rho}.$$

The adiabatic radial problem can therefore be described by the differential equations

$$\begin{aligned}
\frac{d}{dr}\left(\frac{\delta r}{r}\right) &= -\frac{1}{r}\left(3\frac{\delta r}{r} + \frac{1}{\Gamma_1}\frac{\delta P}{P}\right) , \\
\frac{d}{dr}\left(\frac{\delta P}{P}\right) &= -\frac{1}{P}\frac{dP}{dr}\left\{\frac{\delta P}{P} + \left(4 - \frac{r^3 s^2}{Gm}\right)\frac{\delta r}{r}\right\} .
\end{aligned}$$

and the boundary conditions

$$\begin{aligned}
3\frac{\delta r}{r} + \frac{1}{\Gamma_1}\frac{\delta P}{P} = 0 \quad \text{at} \quad r = 0 \, , \\
\frac{\delta P}{P} + \left(4 - \frac{R^3 s^2}{GM}\right)\frac{\delta r}{r} = 0 \quad \text{at} \quad r = R \, .
\end{aligned}$$

It is useful to use dimensionless variables. We define

$$x = \frac{r}{R}, \quad q = \frac{m}{M}, \quad \xi = \frac{\delta r}{r}, \quad \eta = \frac{\delta P}{P} \quad \text{and} \quad s = -i\sigma \quad \text{with} \quad \sigma = \sqrt{\frac{GM}{R^3}}\omega \,.$$

We have then

$$\begin{aligned} \frac{d\xi}{dx} &= -\frac{1}{x}\left(3\xi + \frac{\eta}{\Gamma_1}\right), \\ \frac{d\eta}{dx} &= -\frac{d\ln P}{dx}\left\{\eta + \left(4 + \frac{x^3\omega^2}{q}\right)\xi\right\}. \end{aligned}$$

with the boundary conditions

$$\begin{aligned} 3\xi + \frac{\eta}{\Gamma_1} = 0 \quad \text{at} \quad x = 0\,, \\ \eta + (4+\omega^2)\xi = 0 \quad \text{at} \quad x = 1\,. \end{aligned}$$

Consider now two homologous stellar configurations. Let $\alpha = R'/R$ and $\beta = M'/M$. We can establish between the points of these two models a correspondence such that

$$r' = \alpha r, \quad m' = \beta m, \quad \rho' = \alpha^{-3}\beta\rho, \quad P' = \alpha^{-4}\beta^2 P, \quad \ldots$$

In such a homologous transformation, the coefficients of the differential system and the boundary conditions are invariant. The two models are characterized by the same values of the dimensionless frequencies

$$\omega' = \omega \qquad \text{and} \qquad \sigma' = \beta^{1/2}\alpha^{-3/2}\sigma\,.$$

The pulsation periods of the two stars are therefore in the same ratio as their dynamical timescales. Indeed,

$$\tau'_{dyn} = \beta^{-1/2}\alpha^{3/2}\tau_{dyn}\,.$$

Eliminating η, we get the differential equation

$$\frac{d}{dr}\left(\Gamma_1 P r^4 \frac{d\xi}{dr}\right) + \left\{r^3 \frac{d}{dr}\left[(3\Gamma_1 - 4)P\right] + \sigma^2 \rho r^4\right\}\xi = 0$$

and the boundary conditions

$$\begin{aligned} \frac{d\xi}{dr} = 0 \quad \text{at} \quad r = 0\,, \\ \Gamma_1 R\frac{d\xi}{dr} + \left(3\Gamma_1 - 4 - \frac{R^3\sigma^2}{GM}\right)\xi = 0 \quad \text{at} \quad r = R\,. \end{aligned}$$

This problem is of the Sturm-Liouville type. Therefore it admits an infinity of eigenvalues which can be labelled such that

$$\sigma_0^2 < \sigma_1^2 < \cdots < \sigma_k^2 < \cdots \quad \text{with} \quad \lim_{k\to\infty}\sigma_k^2 = +\infty\,.$$

The eigenfunction ξ_k associated to σ_k has k zeros in the $]0, R[$ interval. It is also possible to show that the ξ_k form a complete set, i.e., a basis in the space of the ξ functions describing a radial perturbation.

We write the differential equation as

$$\mathcal{L}\xi = \sigma^2 \xi$$

with

$$\mathcal{L}\xi = -\frac{1}{\rho r^4}\frac{d}{dr}\left(\Gamma_1 P r^4 \frac{d\xi}{dr}\right) - \frac{1}{\rho r}\frac{d}{dr}\left[(3\Gamma_1 - 4)P\right]\xi \,.$$

This operator is essentially the same as the one we have introduced earlier (the sign has changed, and it acts on $\delta r/r$ instead of $\overrightarrow{\delta r}$). As exercise, one can verify that it is hermitian for the scalar product

$$(u, v) = \int \rho r^4 u \bar{v}\, dr \,.$$

The σ_k^2 are the stationary values for the functional

$$\Lambda(u) = \frac{(u, \mathcal{L}u)}{(u, u)} \,.$$

In particular

$$\sigma_0^2 = \min_u \frac{(u, \mathcal{L}u)}{(u, u)} \,.$$

If the model is dynamically stable, the operator $\mathcal{L}$ is positive definite and vice-versa.

Integrating by parts, one can write

$$(u, \mathcal{L}u) = \int \left\{ \Gamma_1 P r^4 \left|\frac{du}{dr}\right|^2 - r^3 |u|^2 \frac{d}{dr}\left[(3\Gamma_1 - 4)P\right] \right\} dr \,.$$

Assume that Γ_1 is constant in the whole star. It is obvious that if $\Gamma_1 > 4/3$, for all $u \neq 0$ we have $(u, \mathcal{L}u) > 0$ and dynamical stability is assured. On the other hand if $\Gamma_1 < 4/3$, it is sufficient to consider $u = \textit{constant}$ to show that $\mathcal{L}$ is not positive definite and that consequently there is dynamical instability. In the case where $\Gamma_1 = 4/3$, the dynamical stability is marginal. The system then admits the solution $\sigma = 0$, $\xi = 1$ and $\eta = -4$, which describes a homologous transformation of the star.

Assume $\Gamma_1 > 4/3$. We have

$$\sigma_0^2 \leq \frac{(u, \mathcal{L}u)}{(u, u)} \,.$$

Taking $u = 1$, we have

$$\sigma_0^2 \leq -\frac{\displaystyle\int r^3 \frac{d}{dr}\left[(3\Gamma_1 - 4)P\right] dr}{\displaystyle\int \rho r^4 dr} = (3\Gamma_1 - 4)\frac{\displaystyle\int \frac{Gm}{r} dm}{\displaystyle\int r^2 dm} = (3\Gamma_1 - 4)\,\frac{GM}{R^3}\,\frac{\displaystyle\int \frac{q\, dq}{x}}{\displaystyle\int x^2 dq} \,.$$

We can integrate the second term by parts to get

$$\sigma_0^2 \leq 3\,\frac{3\Gamma_1 - 4}{\Gamma 1}\,\frac{\int c^2 dm}{\int r^2 dm} = 3\,\frac{3\Gamma_1 - 4}{\Gamma_1}\,\frac{\overline{c^2}}{\overline{r^2}}\,.$$

We can find a lower limit for σ_0^2 as follows.

$$\begin{aligned}(u, \mathcal{L}u) &= \int \Gamma_1 P r^4 \left|\frac{du}{dr}\right|^2 dr + (3\Gamma_1 - 4)\int Gm\rho r |u|^2 dr \\ &\geq (3\Gamma_1 - 4)\frac{GM}{R^3}\int \frac{q}{x^3}\rho r^4 |u|^2 dr \geq (3\Gamma_1 - 4)\frac{GM}{R^3}\int \rho r^4 |u|^2 dr\,,\end{aligned}$$

because $q/x^3 = \bar{\rho}(r)/\bar{\rho} > 1$. We then have

$$\sigma_0^2 = \min_u \frac{(u, \mathcal{L}u)}{(u, u)} \geq (3\Gamma_1 - 4)\frac{GM}{R^3}\,.$$

If Γ_1 is not constant, it is no longer possible to take it out of the integration sign, unless we use some average value $\overline{\Gamma}_1$. Γ_1 can then be smaller than 4/3 in a limited region of the star without leading to radial dynamical instability.

8.1 Energy of a mode

We write the momentum equation as

$$\rho\frac{d^2\delta r}{dt^2} = \frac{4Gm\rho}{r^3}\delta r - \frac{\partial\,\delta P}{\partial r}\,,$$

and we multiply it by $d\,\delta r/dt$

$$\rho\frac{d}{dt}(\frac{1}{2}v^2 - \frac{2Gm}{r^3}\delta r^2) = -\frac{\partial\,\delta P}{\partial r}\frac{d\,\delta r}{dt}\,.$$

We transform the right-hand side of this equation

$$-\frac{\partial\,\delta P}{\delta r}\frac{d\,\delta r}{dt} = -\vec{v}\cdot \mathrm{grad}\,\delta P = -div(\vec{v}\,\delta P) + \delta P\,\mathrm{div}\,\vec{v}\,,$$

then

$$\delta P\,\mathrm{div}\,\vec{v} = -\frac{\Gamma_1 P}{\rho}\delta\rho\frac{d}{dt}\frac{\delta\rho}{\rho} = -\frac{1}{2}\rho c^2\frac{d}{dt}\left(\frac{\delta\rho}{\rho}\right)^2$$

and finally

$$\frac{d}{dt}\Big\{\rho\Big[\frac{1}{2}v^2 + \frac{1}{2}c^2\Big(\frac{\delta\rho}{\rho}\Big)^2 - 2\frac{Gm}{r^3}(\delta r)^2\Big]\Big\} = -\,\mathrm{div}(\delta P\,\vec{v})$$

or

$$\frac{d}{dt}(\rho\mathcal{E}) = -\,\mathrm{div}\,\overrightarrow{\mathcal{F}}\,.$$

$$\mathcal{E} = \frac{1}{2}v^2 + \frac{1}{2}c^2\Big(\frac{\delta\rho}{\rho}\Big)^2 - 2\frac{Gm}{r}\Big(\frac{\delta r}{r}\Big)^2$$

is the mechanical energy of the pulsation per unit mass. The three terms correspond to the kinetic energy, acoustic potential energy, and gravific potential energy respectively.

The vector

$$\vec{\mathcal{F}} = \delta P\,\vec{v}$$

is the energy flux density.

We group together the two potential energy terms

$$\mathcal{E}_P = \mathcal{E}_A + \mathcal{E}_G\,.$$

We then have

$$\mathcal{E} = \mathcal{E}_K + \mathcal{E}_P\,.$$

If we isolate the time-dependence as

$$\delta r(r,t) = \delta r(r)\,\cos\sigma t\,,$$

we see that

$$\mathcal{E}_K(r,t) = \mathcal{E}_K(r)\sin^2\sigma t \qquad \text{and} \qquad \mathcal{E}_P(r,t) = \mathcal{E}_P(r)\cos^2\sigma t\,.$$

Integrating these expressions over the whole stellar mass we have

$$\begin{aligned} E_K(t) &= \int \mathcal{E}_K(r,t)\,dm = E_K\sin^2\sigma t\,,\\ E_P(t) &= \int \mathcal{E}_P(r,t)\,dm = E_P\cos^2\sigma t\,,\\ E(t) &= \int \mathcal{E}(r,t)\,dm = E_K\sin^2\sigma t + E_P\cos^2\sigma t\,. \end{aligned}$$

Integrating the equation of conservation of the mechanical energy of the pulsation over the whole volume of the star we get

$$\frac{dE}{dt} = 0 \qquad \text{i.e.} \qquad E = constant\,.$$

It follows that

$$E = E_K = E_P$$

and

$$\overline{E_K(t)} = \overline{E_P(t)} = \frac{1}{2}E.$$

The total pulsation energy can be written as

$$E = \frac{\sigma^2}{2}\int \delta r^2\,dm$$

where the δr under the integration sign is the time-independent amplitude $\delta r(r)$.

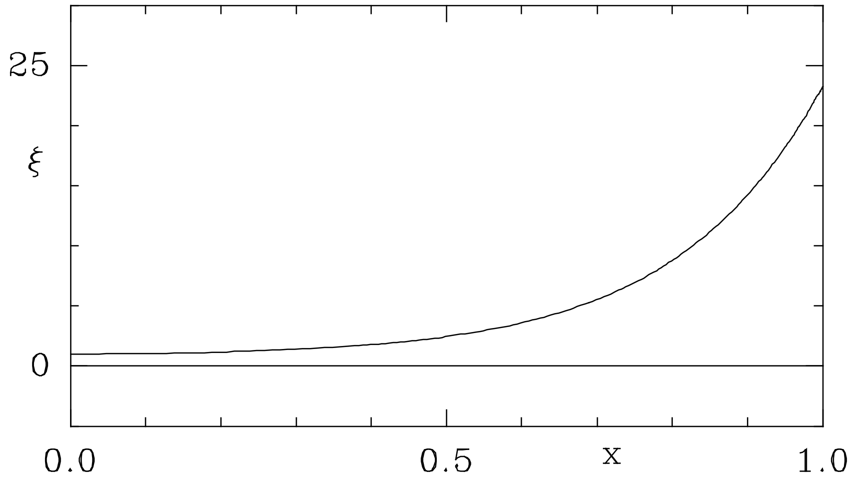

Figure 8.1: Fundamental radial oscillation mode of the standard model (polytrope of index 3).

8.2 Behavior of the eigenfunctions

It is important to have an idea of the behavior of the eigenfunctions ξ_k for a dynamically stable model. The figures 8.1 to 8.3 show some radial oscillations eigenfunctions for the standard model (polytrope of index 3). The arbitrary factors in their definition were chosen such that

$$(\xi_0, \xi_0) = (\xi_1, \xi_1) = \cdots$$

in order to allow a comparison between modes.

We note that the amplitude increases in magnitude as we get near the surface, and it does it faster as the mode order is higher. This is in complete agreement with the fact that at the surface we have

$$\frac{d \ln |\xi|}{dx} = \frac{\omega^2 + 4 - 3\Gamma_1}{\Gamma_1} > 0 \, .$$

We define the average of a physical quantity X weighted by the energy of the k-th mode as

$$\langle X \rangle = \frac{\displaystyle\int X r^2 \xi_k^2 \, dm}{\displaystyle\int r^2 \xi_k^2 \, dm} \, .$$

The value of $\langle x \rangle$ $(x = r/R)$ allows then to measure the relative importance of the central regions and the external layers for the considered mode. The following table gives a few indications for some radial modes of the standard model.

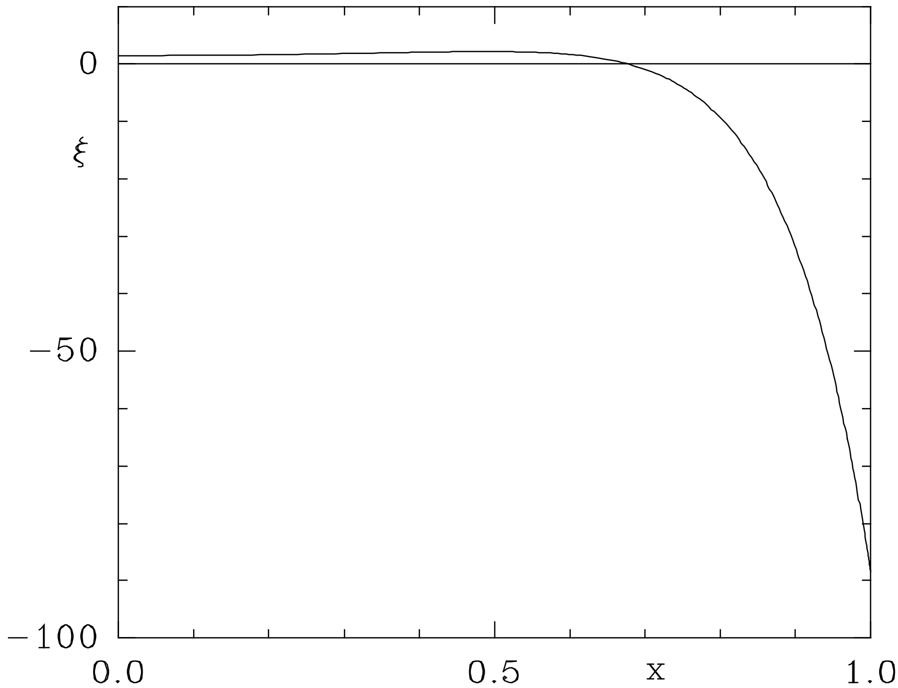

Figure 8.2: First harmonic of the radial oscillation of the standard model (polytrope of index 3).

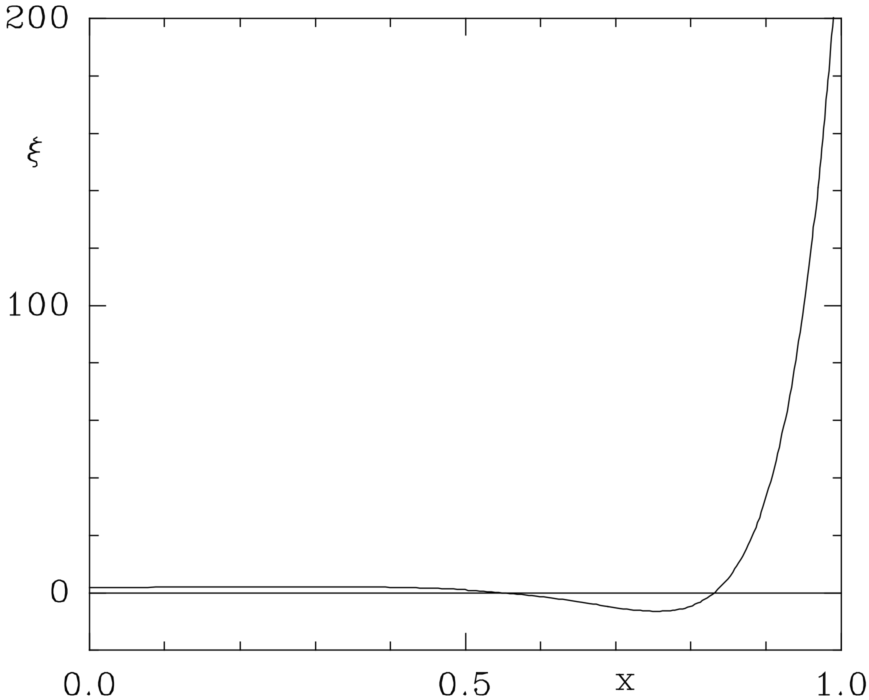

Figure 8.3: Second harmonic of the radial oscillation of the standard model (polytrope of index 3).

Mode	ω	ξ_s/ξ_c	$\langle x \rangle$
0	3.04215	22.44	0.633
1	4.12123	-58.88	0.681
2	5.33690	135.3	0.715
...	...	...	...
9	14.16770	-1681	0.735

We see from this table that the external layers play a more important role for the harmonics than for the fundamental mode, and this role increases as the order of the mode increases. This is true not only for the standard model, but is in fact completely general.

Let us also point out that the amplitude ratio between the surface and the center is higher if the model concentration ($\rho_c/\overline{\rho}$) is higher.

8.3 Some unstable cases

At the border between dynamical stability and instability, the characteristic timescale of the unstable mode stops being small compared to the characteristic timescales of the secular modes. In that case, it is artificial to distinguish between dynamical and secular instability

1) In supermassive stars ($M > 10^5 M_\odot$), the radiation pressure is larger than gas pressure. Indeed the ratio β of the gas pressure to the total pressure is approximately given by

$$\beta\mu = 4.28\sqrt{\frac{M_\odot}{M}}\,.$$

However when β is small, Γ_1 is close to 4/3 and is given approximately by

$$\Gamma_1 \approx \frac{4}{3} + \frac{\beta}{6}\,.$$

For such stars it is necessary to consider general relativity effects, and the criterion for dynamical stability can be written as

$$\Gamma_1 > \frac{4}{3} + 2.25\frac{GM}{Rc^2} \qquad (c = \text{speed of light})\,.$$

As a result, the supermassive stars whose mass is larger than some critical mass between 10^5 and 10^6 $M_\odot$ are dynamically unstable.

2) Several effects tend to dynamically destabilize very condensed white dwarfs. The relativistic degeneracy of the electrons lowers Γ_1 around 4/3. On the other hand, the equilibrium between β disintegration and electron capture contributes also to lower Γ_1. Finally the general relativity effects make the stability criterion harder to satisfy. The configurations with a central density larger than a critical density of the order of 10^{10} g cm^{-3} are dynamically unstable. The only stable high density stars are the neutron stars.

3) In the initial contraction phases of a proto-star, there is no dynamically stable equilibrium model due to the H_2 dissociation and the ionization of H and He. In these phases the proto-star therefore evolves on the dynamical timescale.

4) In the final phases of evolution of a sufficiently massive star, the establishment of a nuclear equilibrium (iron-peak) lowers Γ_1 and creates a dynamical instability. This instability will define the initial phase of a supernova.

5) The enormous mass losses of the S Dor (or LBV) variable stars could be due to a dynamical instability in their envelope. In these massive and very luminous stars, the radiation pressure is very high (β small) and the existence of a hydrogen and helium partial ionization zone is enough to lower the average value of Γ_1 below the critical value.

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The reader is refered to the book by Ince (1956), chapter 10, for more details on the Sturm-Liouville problem, and to the article by Beyer (1995) for a rigorous proof of the properties of the spectrum of adiabatic radial oscillations.

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Chapter 9

Asymptotic expression for the radial frequencies

We have seen above that the differential equation describing the adiabatic radial oscillations can be written as

$$\frac{d}{dr}\left(\Gamma_1 P r^4 \frac{d\xi}{dr}\right) + \left\{ r^3 \frac{d}{dr}[(3\Gamma_1 - 4)P] + \sigma^2 \rho r^4 \right\} \xi = 0.$$

For high order modes, the σ^2 term is the dominant term in the coefficient of ξ. This will allow the use of asymptotic methods to obtain approximate values for the oscillation frequencies. There are different such methods, more or less complicated, and more or less rigorous. We will restrict ourselves to a simple approach.

We use the change of variables

$$\tau = \int_0^r \frac{dr}{c} \qquad \text{and} \qquad w = r^2 (\Gamma_1 P \rho)^{1/4} \xi.$$

We note that τ is the time needed to travel, at the sound speed, from the center of the star to the point under consideration. The differential equation reduces to

$$\frac{d^2 w}{d\tau^2} + \left\{ \sigma^2 + \frac{1}{r(\Gamma_1 P \rho)^{1/2}} \frac{d}{d\tau}[(3\Gamma_1 - 4)P] - \frac{1}{r^2 (\Gamma_1 P \rho)^{1/4}} \frac{d^2}{d\tau^2}[r^2 (\Gamma_1 P \rho)^{1/4}] \right\} w = 0.$$

The solution must satisfy the boundary conditions $w(0) = w(\tau_R) = 0$, where $\tau_R = \tau(R)$.

For high order modes, σ^2 is large and it is tempting to ignore the other terms in the coefficient w. However, these terms are singular at the center and at the surface, and they are therefore not negligible compared to σ^2 near those points. Without these singularities we could simplify the equation to

$$\frac{d^2 w}{d\tau^2} + \sigma^2 w = 0.$$

The solutions satisfying the boundary conditions have the form

$$w_k \propto \sin \sigma_k \tau \qquad \text{with} \qquad \sigma_k = \frac{k\pi}{\tau_R} \qquad \text{for} \quad k = 1,\, 2,\, 3,\, \ldots$$

w_k has $k-1$ nodes in the interval $]0,\ \tau_R[$. If we number the modes like this, we must give the value $k=1$ to the fundamental mode, $k=2$ to the first harmonic, ... We will justify this numbering of the modes when we will study the non radial modes.

To take into effect the singularities, we will develop two approximations. The first one will take into account the central singularity, and the second the surface singularity. We'll join the two solutions at a point situated far from both the center and the surface.

9.1 Central singularity

The singularity comes from the term with a second derivative with respect to τ. We can write the differential equation as

$$\frac{d^2w}{d\tau^2}+\left[\sigma^2-\frac{2}{\tau^2}+f(\tau)\right]w=0,$$

where $f(\tau)$ is a regular function of τ at $\tau=0$. We can now neglect $f(\tau)$ compared to the term in σ^2 and the singular term. The equation can be reduced to a Bessel equation by writing $z=\sigma\tau$ and $w=\sqrt{z}u(z)$.

$$\frac{d^2u}{dz^2}+\frac{1}{z}\frac{du}{dz}+\left(1-\frac{9}{4z^2}\right)u=0.$$

The regular solution at $z=0$ is given by the Bessel function of the first kind of order 3/2, $u(z)=J_{3/2}(z)$. Let $w'(\tau)$ be the approximative solution thus obtained. It is valid as long as we are not too close to the surface, whose singularity has not been taken into account. We use the asymptotic approximation of $J_{3/2}(z)$ for large z to write $w'(\tau)$, not too close to the center, as

$$w'(\tau)\propto\sin(\sigma\tau-\frac{\pi}{2}).$$

9.2 Surface singularity

We can roughly describe the structure of the superficial layers with an effective polytropic index n_e such that $\rho\propto(R-r)^{n_e}$ and $P\propto(R-r)^{n_e+1}$. If these layers are convective, n_e is equal to the polytropic index associated to the value of Γ_1, i.e. $n_e=1/(\Gamma_1-1)$. If the superficial layers are radiative and if the opacity is given by $\kappa\propto\rho^rT^{-s}$, n_e is given by $n_e=(r+3)/(s+1)$. We easily get the following relations

$$\tau_R-\tau\propto(R-r)^{1/2},\qquad\rho\propto(\tau_R-\tau)^{2n_e},\qquad P\propto(\tau_R-\tau)^{2n_e+1},\qquad c\propto(\tau_R-\tau).$$

We can then write the differential equation as

$$\frac{d^2w}{d\tau^2}+\left[\sigma^2-\frac{n_e^2-\frac{1}{4}}{(\tau_R-\tau)^2}+f(\tau)\right]w=0,$$

where $f(\tau)$ is a regular function in τ_R. Omitting this term and using $z = \sigma(\tau_R - \tau)$ and $w = \sqrt{z}u(z)$, we get

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{n_e^2}{z^2}\right) u = 0,$$

whose regular solution is given by $J_{n_e}(z)$. We thus obtain an approximation $w''(\tau)$ valid away from the center. Using the asymptotic expression of the Bessel function for large z, we can write for $w''(\tau)$, not too close to the surface,

$$w''(\tau) \propto \sin(\sigma\tau - \sigma\tau_R - \frac{\pi}{4} + \frac{n_e\pi}{2}).$$

9.3 Joining of the two solutions

We join the two solutions $w'(\tau)$ and $w''(\tau)$ at any point where both approximations are valid. It is enough to ask that the phase difference of the sine arguments be a multiple of π and we then have

$$w_k(\tau) \propto \sin(\sigma_k\tau - \frac{\pi}{2}) \qquad \text{with} \qquad \sigma_k = (k + \frac{n_e}{2} + \frac{1}{4})\frac{\pi}{\tau_R} \qquad \text{for} \quad k = 1,\, 2,\, 3,\, \ldots$$

We check that $w_k(\tau)$ has indeed $k-1$ zeros in the interval $]0,\, \tau_R[$. Let τ^* be one zero of w_k. We'll assume that it is the ℓ-th from the center and the m-th from the surface; w_k then has $\ell + m - 1$ zeros. Using the asymptotic expression for the zeros of $J_\nu(z)$, we get

$$\sigma_k\tau^* = (\ell + \frac{1}{2})\pi \qquad \text{and} \qquad \sigma_k(\tau_R - \tau^*) = (m + \frac{n_e}{2} - \frac{1}{4})\pi.$$

Adding these two relations and using the expression for σ_k, we get $k = \ell + m$, which confirms that w_k indeed has $k-1$ zeros. The modes are therefore numbered by the index k as above, the index 1 being attributed to the fundamental mode, 2 to the first harmonic, ...

9.4 Brief reminder on Bessel functions

The regular solution at $z = 0$ of the Bessel equation

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) u = 0$$

is the Bessel function of the first kind of order ν, written $J_\nu(z)$. For large positive values of z, it can be approximated by

$$J_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \sin\left(z + \frac{\pi}{4} - \frac{\nu\pi}{2}\right)$$

and its k-th positive zero $j_{\nu,k}$, for large values of k, is approximately given by

$$j_{\nu,k} \approx \left(k + \frac{\nu}{2} - \frac{1}{4}\right)\pi.$$

References

The asymptotic behavior of the adiabatic radial oscillations has been studied by Ledoux (1962, 1963) using the Langer (1935) method. We have essentially followed that method here. Tassoul and Tassoul (1968) have obtained a more precise approximation (of the second order) using Olver's (1956) method. Through a different choice of the large parameter, Ruymaekers and Smeyers (1991) have improved the above approximations.

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Chapter 10

Vibrational stability

Consider a dynamically stable stellar model. We can calculate the frequencies of the normal modes of oscillation with the adiabatic approximation, but we cannot determine whether the modes are excited or damped by thermal processes. We would also like to know what are the excitation mechanisms of a mode observed in a variable star. It is necessary to take into account the non adiabatic terms and to solve the fourth order differential system to get this kind of information.

We have already established the relation

$$s^2 \int |\delta r|^2 dm + \int \Big\{ \frac{\delta P}{\rho} \frac{\overline{\delta\rho}}{\rho} + \frac{4r}{\rho} \frac{dP}{dr} \left| \frac{\delta r}{r} \right|^2 \Big\} dm = 0.$$

We can take the imaginary part of it

$$2\,\Re s\,\Im s = -\frac{\Im \displaystyle\int \frac{\delta P}{\rho} \frac{\overline{\delta\rho}}{\rho}\, dm}{\displaystyle\int |\delta r|^2 dm}.$$

We have also shown that the expression in the denominator is related to the mechanical energy of the oscillation. There is also a simple physical interpretation for the numerator. Consider a gram of matter, undergoing the thermodynamical cycle of period τ described by the equations

$$\begin{aligned} P(t) &= P_0 + a\cos(\phi - \sigma t), \\ \rho(t) &= \rho_0 + b\cos(\psi - \sigma t). \end{aligned}$$

Using the usual conventions, this can also be written as

$$\begin{aligned} \delta P(t) &= \delta P\, e^{-i\sigma t} \qquad (\delta P = a e^{i\phi}), \\ \delta\rho(t) &= \delta\rho\, e^{-i\sigma t} \qquad (\delta\rho = b e^{i\psi}). \end{aligned}$$

The work done by this system during one cycle can be written as

$$\begin{aligned} \mathcal{T} &= \oint P\, dV = \int_0^\tau P \frac{dV}{dt} dt = -\int_0^\tau \frac{P}{\rho^2} \frac{d\rho}{dt} dt \\ &= \frac{\pi a b}{\rho^2} \sin(\phi - \psi) = \pi \Im\Big(\frac{\delta P}{\rho} \frac{\overline{\delta\rho}}{\rho} \Big). \end{aligned}$$

The average power of the thermodynamical cycle can then be written as

$$\mathcal{W} = \frac{\mathcal{T}}{\tau} = \frac{\sigma}{2}\Im\Big(\frac{\delta P}{\rho}\frac{\overline{\delta\rho}}{\rho}\Big).$$

Let $W = \int \mathcal{W}\, dm$ be the average power developed by the whole star . We have

$$\Im \int \frac{\delta P}{\rho}\frac{\overline{\delta\rho}}{\rho} dm = \frac{2}{\sigma} W.$$

We can transform the numerator as done above to obtain the cubic equation in s. More simply, we can use the cubic equation

$$s^2 + A + \frac{1}{s}B = 0.$$

We take its imaginary part, and we get

$$2\,\Re s\,\Im s = -\frac{\Im\frac{1}{s}\int(\Gamma_3 - 1)\frac{\overline{\delta\rho}}{\rho}\Big(\delta\epsilon - \frac{d\,\delta L}{dm}\Big)dm}{\int |\delta r|^2 dm}$$

10.1 The quasi-adiabatic approximation

The adiabatic approximation is usually excellent in the internal layers of the star and fails only in its external layers. Therefore we consider the non conservative terms as small perturbations to the adiabatic case. We can then obtain some corrections to the eigenfunction and to the eigenvalue s using a perturbation method. Since we are mostly interested in knowing whether a given mode is excited or damped by the non conservative terms, we are interested in the real part of s. It can be obtained simply from the above expression, by substituting the adiabatic solutions in the right-hand side.

We write $s = \sigma' - i\sigma$, where σ' will be called the growth coefficient (or $-\sigma'$ the damping coefficient). We have

$$\sigma' = \frac{1}{2\sigma^2}\frac{\int \frac{\delta T}{T}\Big(\delta\epsilon - \frac{d\,\delta L}{dm}\Big)dm}{\int |\delta r|^2 dm}\,.$$

There is a simple physical explanation for this expression. We recall that

$$W = \frac{\sigma}{2}\Im\int\frac{\delta P}{\rho}\frac{\overline{\delta\rho}}{\rho}dm = \frac{1}{2}\int\frac{\delta T}{T}\Big(\delta\epsilon - \frac{d\,\delta L}{dm}\Big)dm\,,$$

$$E = \frac{\sigma^2}{2}\int |\delta r|^2 dm\,,$$

so that we get

$$\sigma' = \frac{W}{2E}\,.$$

This result can be justified as follows. If the amplitude of the oscillation grows exponentially as $e^{\sigma' t}$, its energy grows as $e^{2\sigma' t}$. We then have

$$2\sigma' = \frac{1}{E}\frac{dE}{dt} = \frac{W}{E}\,.$$

An obvious benefit of the integral expression of the growth coefficient is the interpretation we have made for its numerator. On one hand, the excitation or damping role of each stellar layer can be assessed. If it brings a positive (negative) contribution to the integral, it has an exciting (damping) role for the oscillation. On the other hand, this expression allows us to determine the mechanism responsible for the excitation or the damping (i.e. whether it is due to the transport or to the energy generation term).

There is however a serious problem with the quasi-adiabatic approximation. The adiabatic eigenfunctions used to calculate the integral are not a valid approximation in the external layers. A transition zone is often defined by the relation

$$c_v T \Delta m \approx L\tau$$

where τ is the period and Δm the mass above the point under consideration. $c_v T \Delta m$ is of the order of the internal energy of the layers above the considered level. $L\tau$ is the energy radiated during one period. We can then separate the star into three regions

1°) an internal adiabatic zone where $c_v T \Delta m \gg L\tau$. The adiabatic approximation is very good here.

2°) the transition zone

3°) an external zone, strongly non adiabatic, where $c_v T \Delta m \ll L\tau$. Here, we will write the thermal energy conservation equation using the following approximate relation

$$\frac{\Delta\,\delta L}{\Delta m} = -s c_v T \frac{\delta S}{c_v}\,.$$

The variation of $\delta L/L$ in this zone can be written as

$$\Delta\frac{\delta L}{L} = -\frac{s c_v T \Delta m}{L}\frac{\delta S}{c_v}\,.$$

The coefficient in the right-hand side is very small. We conclude that the heat capacity of these superficial layers is too small to influence δL, which is more or less constant through the whole zone. One must be careful however when an abundant element is partially ionized, because the ionization process can still absorb or liberate large quantities of energy.

Due to its small heat capacity, the strongly non adiabatic zone cannot strongly affect the excitation or the damping of the oscillation. However, the δL calculated using the adiabatic approximation grows in the external layers of the star. The disagreement between the true behavior of δL and the one deduced from the adiabatic approximation can badly influence the result of the quasi-adiabatic calculation. The problem is often solved by excluding the strongly non adiabatic zone from the integration domain.

$$\int_0^M \frac{\delta T}{T}\Big(\delta\epsilon - \frac{d\,\delta L}{dm}\Big)dm \longrightarrow \int_0^{m^*} \frac{\delta T}{T}\Big(\delta\epsilon - \frac{d\,\delta L}{dm}\Big)dm\,.$$

The following equality is often used as a cutoff criterion for the integration

$$\left|\frac{\delta T}{T}\right|_{ad} = \left|\frac{\delta T}{T}\right|_{nonad}$$

where

$$\left|\frac{\delta T}{T}\right|_{ad} = (\Gamma_3 - 1)\left|\frac{\delta\rho}{\rho}\right| \quad \text{and} \quad \left|\frac{\delta T}{T}\right|_{nonad} = \left|\frac{\delta S}{c_v}\right| = \frac{1}{\sigma c_v T}\left|\frac{d\,\delta L}{dm}\right|.$$

This process is quite crude. When the excitation mechanism lies in the central regions of the star, it is legitimate to hope that the result will be meaningful. The situation is more delicate when the excitation mechanism lies in the external layers of the star (for example in the transition zone). It is then more prudent to integrate the whole system of equations rather than to use the quasi-adiabatic approximation.

When we integrate directly the non adiabatic equations the coefficient of $\delta S/c_v$ in the thermal energy equation is very large in the internal layers of the star, and this can create some problems for the numerical integration. They can however be solved. Another difficulty comes from the fact that σ' is small compared to σ, which makes it hard to evaluate precisely the growth coefficient. In this case it is possible to improve the precision of σ' by using the integral expression with the eigenfunctions of the non adiabatic case.

10.2 The nuclear excitation

The nuclear term always has an exciting influence on the pulsation. The nuclear energy generation generally happens in the internal layers of the star, where the adiabatic approximation is excellent. The nuclear contribution to the numerator of the growth coefficient can be written as

$$\int \frac{\delta T}{T}\delta\epsilon\, dm = \int (\Gamma_3 - 1)[\epsilon_\rho + (\Gamma_3 - 1)\epsilon_T]\Big(\frac{\delta\rho}{\rho}\Big)^2 \epsilon\, dm > 0\,.$$

For main sequence stars, ϵ decreases very rapidly from the center. Only the central regions will participate in this destabilizing effect. We have seen that the eigenfunctions are usually small in the central regions. This is unfavorable to the exciting influence of the nuclear reactions, which is in competition with the generally damping effect of the transport terms (see later).

The excitation mechanism relying on the nuclear energy generation is called the ϵ mechanism.

For massive main sequence stars, the radiation pressure is larger and Γ_1 is close to 4/3. As a result, the eigenfunctions grow at a slower rate towards the exterior (for $\Gamma_1 = 4/3$, we have $\delta r/r$ = constant). This case is more favorable to the development of a vibrational instability of nuclear origin. Calculations indeed show that main sequence stars with a mass higher than some critical mass are vibrationnally unstable. The exact value of the critical mass depends on the chemical composition and on the opacity and the energy generation laws used. It goes from 90 $M_\odot$ for metal-poor stars to 120–150 $M_\odot$ for population I stars (Stothers, 1992). These stars however suffer from a more violent instability associated to the strange modes (see later).

10.3 The influence of the transport terms

The contribution of the transport term to the numerator of the expression for σ' is given by

$$-\int \frac{\delta T}{T} \frac{d\,\delta L}{dm}\, dm = -\int \frac{\delta T}{T} \frac{d\,\delta L}{dr}\, dr\,.$$

$-\dfrac{\delta T}{T}\dfrac{d\,\delta L}{dm}$ is positive (destabilizing effect) if δL decreases towards the exterior when $\delta T > 0$, i.e. if the matter absorbs the heat at high temperature and gives it back at low temperature. This is the usual mechanism of a thermodynamical engine.

Given the rapid growth of the eigenfunctions towards the exterior, it is the exterior layers which will contribute the most to the integral and more exactly the transition zone. We recall however that δL tends to be constant in the exterior layers (unless an abundant element is partially ionized).

Let's try to estimate the sign of the contribution of the transport term, where the adiabatic approximation is still valid. We have

$$\frac{\delta T}{T} \approx (\Gamma_3 - 1)\frac{\delta\rho}{\rho}$$

and we get from the transport equation

$$\frac{\delta L}{L} = \frac{\dfrac{d(\delta T/T)}{dr}}{\dfrac{d\ln T}{dr}} + 4\frac{\delta r}{r} + 4\frac{\delta T}{T} - \frac{\delta\kappa}{\kappa}\,.$$

For low order modes we can neglect the $\delta T/T$ derivative term and the $\delta r/r$ term to get

$$\frac{\delta L}{L} \approx \left[(4-\kappa_T)(\Gamma_3 - 1) - \kappa_\rho\right] \frac{\delta\rho}{\rho}\,.$$

In usual circumstances (e.g., Kramers opacity law) we have

$$\Gamma_3 \approx 5/3\,, \quad \kappa_\rho \approx 1\,, \quad \kappa_T \approx -3.5\,.$$

Therefore

$$\left[(4-\kappa_T)(\Gamma_3 - 1) - \kappa_\rho\right] \approx 4 > 0\,.$$

$\delta L/L$ has thus the same sign than $\delta\rho/\rho$ and $\delta T/T$ and grows in magnitude. As a result

$$-\frac{\delta T}{T}\frac{d\,\delta L}{dr} < 0\,,$$

which shows that the transport terms generally have a stabilizing effect.

To have an exciting effect, it is neccessary to change the sign of the coefficient for the transport terms. This can happen if one of the following conditions is satisfied.

1°) If $\Gamma_3 - 1$ is small enough. This can happen in a region where an abundant element is partially ionized.

Table 10.1: Ratio between the growth time and the period for some variables.

Type of variable	τ'/τ
Classical Cepheids and RR Lyrae	10^2 to 10^3
δ Sct	10^4 to 10^6
W Vir	10 to 20
long period variables (Mira)	1 to 10

2°) If κ_T is positive, which happens in the external layers, due to the presence of the ion H^-.

This excitation mechanism which relies on the increase of the opacity during an adiabatic compression is called the κ mechanism. It is also sometimes called the γ mechanism, when one wants to insist on the role played by the decrease of $\Gamma_3 - 1$ in the exciting region.

The variability of some intrinsically variable stars can be explained by a vibrational instability due to the transport terms and resulting from the partial ionization of an abundant element. The mechanism lies in the partial second ionization zone of helium ($He^+ \rightleftharpoons He^{++}$) for variables of the instability strip: RR Lyr, δ Cep, W Vir, RV Tau, δ Sct. In the case of Mira-type variables , the partial ionization of hydrogen ($H \rightleftharpoons H^+$) could be responsible for the instability, and in the case of β Cep type variables, the cause could be an increase of the opacity due to iron around 200000 K.

Table 1 gives the order of magnitude of the ratio between the growth time $\tau' = 1/\sigma'$ and the period τ.

10.4 Strange modes

This label designates dynamical modes initially discovered in very luminous stellar models and whose behavior was very puzzling. Thus, in very massive (50 to 150 $M_\odot$) main sequence models, the frequencies of the strange modes evolve differently from those of the other modes when the stellar mass is changed. At the points where the strange modes frequencies should cross the frequencies of the regular modes, we see either avoided crossing or the development of an instability (the two modes whose frequencies are crossing acquire $\Re s$ of opposite signs). In many cases, the strange modes are strongly non adiabatic and their existence seems to be linked to a much larger radiation pressure than gas pressure and to a density inversion in an external convective zone. The pulsation energy is confined in a small region including and above the density inversion (trapped mode). In massive main sequence stars these modes are responsible for the instability for masses larger than a critical mass going from 60 $M_\odot$ for $Z = 0.03$ to more than 120 $M_\odot$ for metal poor stars.

These strange modes could also be responsible for violent instabilities in LBV variables. They were also found in hydrogen deficient stars, in low-mass supergiants, in the stars at the center of planetary nebulae, in Wolf-Rayet stars, and even in cepheids.

Buchler et al. (1997) showed that strange modes exist in weakly non adiabatic stars (cepheids) and that their behavior can be explained in simple mechanical terms. The equations for the pulsation can be written under a form similar to the Schrödinger equation. The stars with strange modes are characterized by the existence of a potential barrier (potential meaning here a term which plays the same role as a potential in the Schrödinger equation) which allows the trapping of the modes (the strange modes) in the external stellar layers. The trapping of the strange modes explains the particularities of their behavior. Another point of view and a detailed bibliography can be found in the paper by Saio et al. (1998).

References

We assumed a model in thermal equilibrium in our discussion. However, thermal equilibrium is not satisfied in stars during some phases of their evolution, and a large fraction of the radiated energy is provided by the gravific contraction. More information on stellar vibrational stability during those evolution phases can be found in Demaret (1974ab, 1975ab, 1976) and Demaret and Perdang (1977).

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Chapter 11

The pulsation mechanism in the instability strip and the light phase lag

Stability calculations of stellar models in the instability strip give results which are more or less in agreement with the observations. They show that, in the instability strip, the fundamental radial mode or the first harmonic are vibrationally unstable. This vibrational instability lies in the second helium ionization zone of the envelope. The hydrogen ionization zone, which roughly coincides with the first helium ionization zone, also contributes, although to a smaller extent, to the instability.

The calculations also predict correctly the position of the blue limit (i.e. the left limit in the HR diagram, on the high temperature side) of the instability strip, but they fail for the red limit. This is due to the fact that when the effective temperature is high, convection transports little energy whereas at lower effective temperatures the transport of energy by convection plays an important role. The lack of a satisfactory theory of convection in the presence of pulsations is probably enough to explain the failure to determine the instability limit on the right side of the HR diagram.

The theory also reproduces the period-luminosity relation obeyed by the cepheid stars and used to estimate stellar distances. This relation is even obeyed by the δ Sct variables as shown in figure 11.1.

11.1 The existence of an instability

Cox (1967) gave a simple interpretation of the instability strip.

The effect of the ionization of an abundant element is to lower $\Gamma_3 - 1$ (figure 11.2) and this is favorable to a vibrational instability because of the transport term. We examine this argument in more details. We have established that in the quasi-adiabatic approximation,

$$\frac{\delta L}{L} \approx [(\Gamma_3 - 1)(4 - \kappa_T) - \kappa_\rho] \frac{\delta \rho}{\rho} .$$

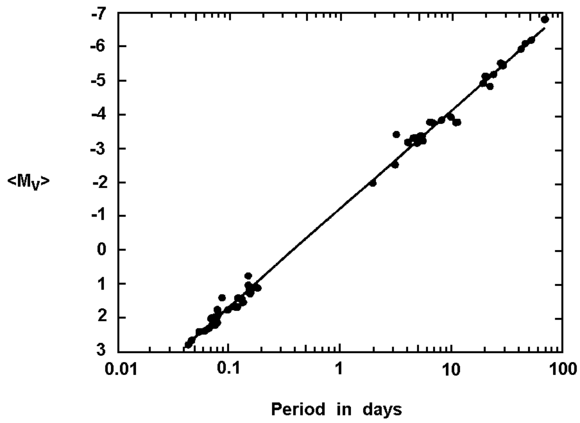

Figure 11.1: Period-luminosity diagram for cepheids and δ Sct variables pulsating in fundamental mode (Fernie, 1992).

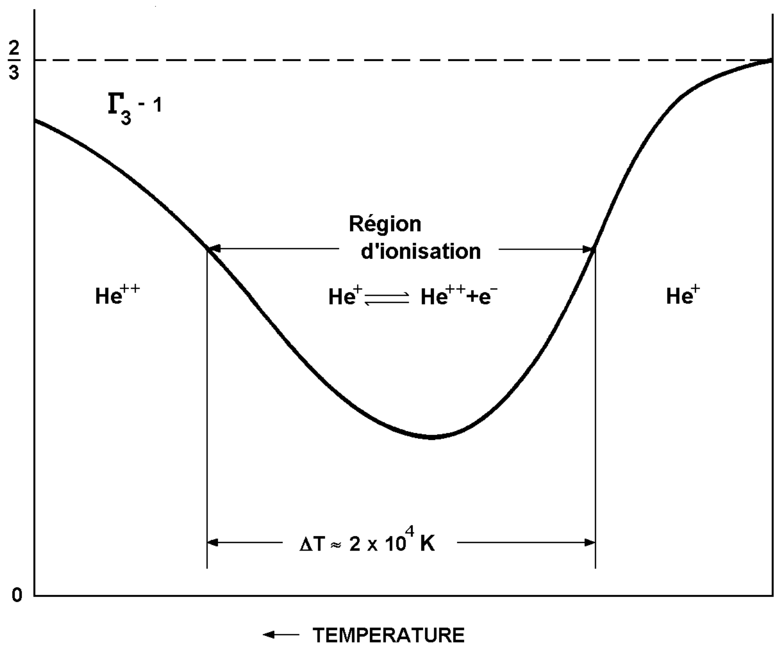

Figure 11.2: Behavior of $\Gamma_3 - 1$ in the ionization region of He^+ in a stellar envelope model (Cox, 1967).

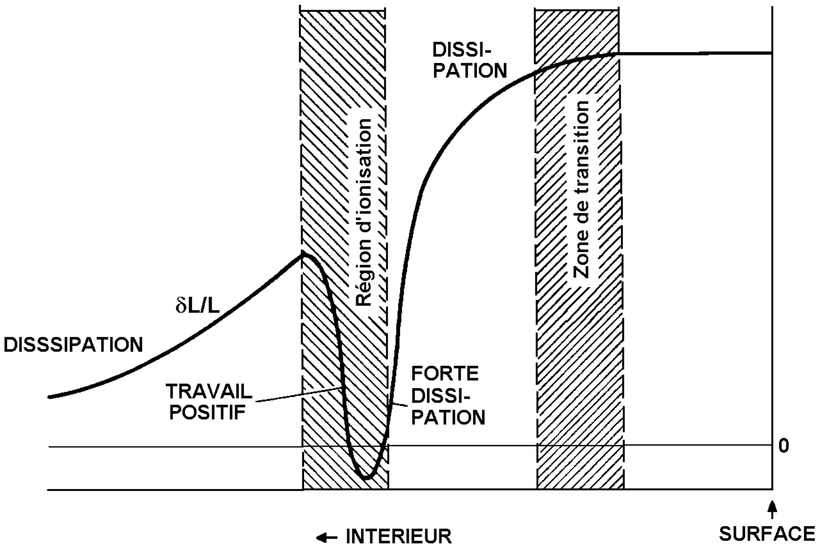

Figure 11.3: Behavior of $\delta L/L$ in the superficial layers of a stellar model (Cox, 1967).

If the ionization zone is within the adiabatic zone of the star, $\delta L/L$ will behave as described in figure 11.3 during compression. In the internal part of the ionization zone δL decreases during compression. This zone therefore absorbs energy at high temperature and its work is positive. This destabilizing effect is balanced in the external part of the ionization zone in which we can carry the opposite reasoning. An ionization zone situated in the adiabatic zone of the star is therefore unable to create a vibrational instability.

An ionization zone in the strongly non-adiabatic zone is also unable to create a vibrational instability because δL is basically constant in these external layers of low heat capacity (figure 11.4).

The most favorable case for the development of the vibrational instability happens when the ionization region coincides with the transition region. If the pulsation ceases to be adiabatic in the external part of the ionization region, δL will tend to become independent of r and the positive work done in the internal part of the ionization region will no longer be compensated. (figure 11.5).

Detailed calculations confirm this interpretation. The instability strip is the region in the HR diagram where the transition region and the helium second ionization region coincide. To the left of this instability strip, for higher effective temperatures, this ionization region lies in the strongly non-adiabatic region. To the right of the instability strip, for lower effective temperatures, this ionization zone lies in the adiabatic region. The hydrogen ionization region now coincides whith the transition zone, but the important role of the convection in the energy transport complicates the process.

11.2 The light phase lag

According to the adiabatic theory, the maximum of the luminosity should correspond to the minimum of the radius. However, for the variables of the instability strip, the

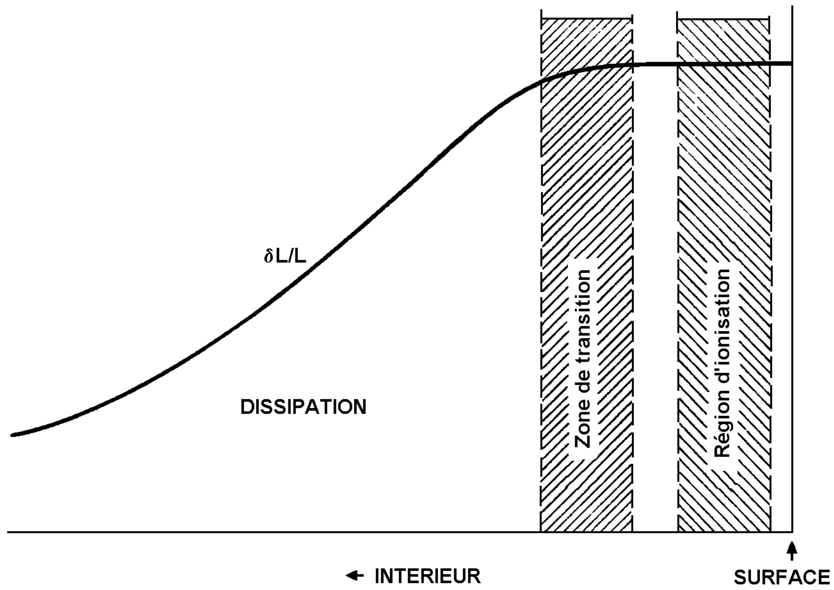

Figure 11.4: Behavior of $\delta L/L$ in the superficial layers of a stellar model (Cox, 1967).

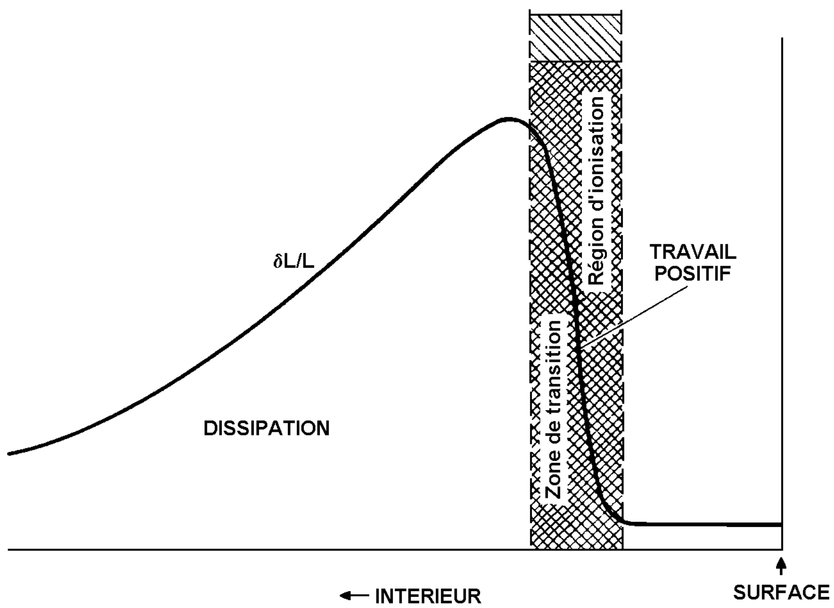

Figure 11.5: Behavior of $\delta L/L$ in the superficial layers of a stellar model (Cox, 1967).

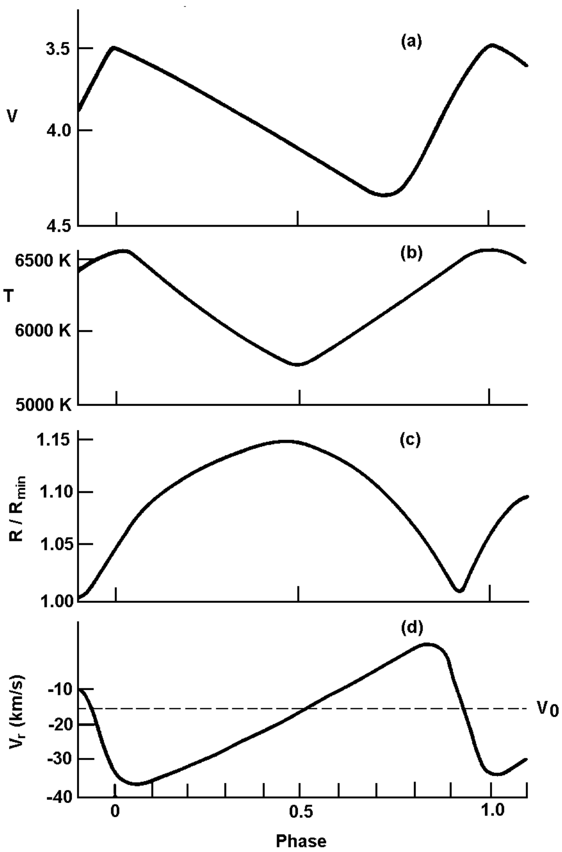

Figure 11.6: The pulsation of δ Cep: (a) light curve, (b) temperature, (c) radius, (d) radial velocity (Petit, 1987).

maximum of the luminosity corresponds to the maximum of the expansion velocity (figure 11.6). For a sine oscillation, the phase lag over the adiabatic prediction should then be of a quarter of the period. It is slightly smaller, because of the asymmetry of the light curves and of the radial velocity. Detailed calculations reproduce nicely this phase lag and show that it is due to the hydrogen ionization zone. It can be explained with a simple linear theory (Castor 1968 et 1971, Cox 1980). We will only describe it briefly here.

Even though the hydrogen ionization zone corresponds to temperatures between 8000 K and 15000 K, it is very narrow and corresponds to a small fraction (of the order a twentieth) of the pressure scale height. This ionization zone can therefore be considered as a discontinuity (as in a phase transition). During the pulsation, this discontinuity moves through the stellar mass. Let assume that below the ionization front, δL and δr have opposite phases. At the minimum of the radius, $\delta L > 0$ below the ionization front. The latter absorbs energy and therefore moves through the stellar mass towards the exterior. It is only a quarter of a period later, when δL goes through 0 below the ionization front, that this one will reach its most exterior position. But the stellar layers above the ionization front have a very simple structure which depends mostly on the position of the ionization front. This is why the effective temperature is in phase with the position of the ionization front and reaches its maximum when the latter reaches its most exterior position.

This mechanism, responsible for the light phase lag, cannot exist in stars whose effective temperature is higher than 10^4K. This is in agreement with the observations: β Cep variables do not exhibit any phase lag.

Note

The ionization potential of H is 13.6 eV. For helium, it is 24.6 eV for the first ionization and 54.4 eV for the second. The second helium ionization zone is around a temperature of 40000 K, while the hydrogen ionization zone is around 10000 K.

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Chapter 12

Non radial oscillations

12.1 Spherical coordinates

The spherical coordinates and the cartesian coordinates are related through:

$$\begin{aligned} x &= r \sin\theta \cos\phi\,, \\ y &= r \sin\theta \sin\phi\,, \\ z &= r \cos\theta\,. \end{aligned}$$

We will use the local cartesian basis $\vec{e}_r$, $\vec{e}_\theta$, $\vec{e}_\phi$. The differential expressions for these vectors are:

$$\begin{aligned} d\vec{e}_r &= \vec{e}_\theta\, d\theta + \sin\theta\, \vec{e}_\phi\, d\phi\,, \\ d\vec{e}_\theta &= -\vec{e}_r\, d\theta + \cos\theta\, \vec{e}_\phi\, d\phi\,, \\ d\vec{e}_\phi &= -(\sin\theta\, \vec{e}_r + \cos\theta\, \vec{e}_\theta)\, d\phi\,. \end{aligned}$$

The expressions for the differential operators applied to the coordinates or the basis vectors can be easily deduced from the preceding relations.

$$\begin{array}{lll} \mathrm{grad}\, r = \vec{e}_r\,, & \mathrm{div}\, \vec{e}_r = \frac{2}{r}\,, & \mathrm{curl}\, \vec{e}_r = 0\,, \\ \mathrm{grad}\, \theta = \frac{1}{r}\vec{e}_\theta\,, & \mathrm{div}\, \vec{e}_\theta = \frac{1}{r}\,\mathrm{cotg}\,\theta\,, & \mathrm{curl}\, \vec{e}_\theta = \frac{1}{r}\vec{e}_\phi\,, \\ \mathrm{grad}\, \phi = \frac{1}{r\sin\theta}\vec{e}_\phi\,, & \mathrm{div}\, \vec{e}_\phi = 0\,, & \mathrm{curl}\, \vec{e}_\phi = \frac{1}{r}\,\mathrm{cotg}\,\theta\, \vec{e}_r - \frac{1}{r}\vec{e}_\theta\,. \end{array}$$

Let α be a scalar field and $\vec{a}$ be a vector field. We will write $\vec{a}$ as

$$\vec{a} = a_r\vec{e}_r + a_\theta\vec{e}_\theta + a_\phi\vec{e}_\phi\,.$$

We easily get

$$\begin{aligned} \mathrm{grad}\, \alpha &= \frac{\partial \alpha}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial \alpha}{\partial \theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial \alpha}{\partial \phi}\vec{e}_\phi\,, \\ \mathrm{div}\, \vec{a} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 a_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(a_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial \phi}\,, \end{aligned}$$

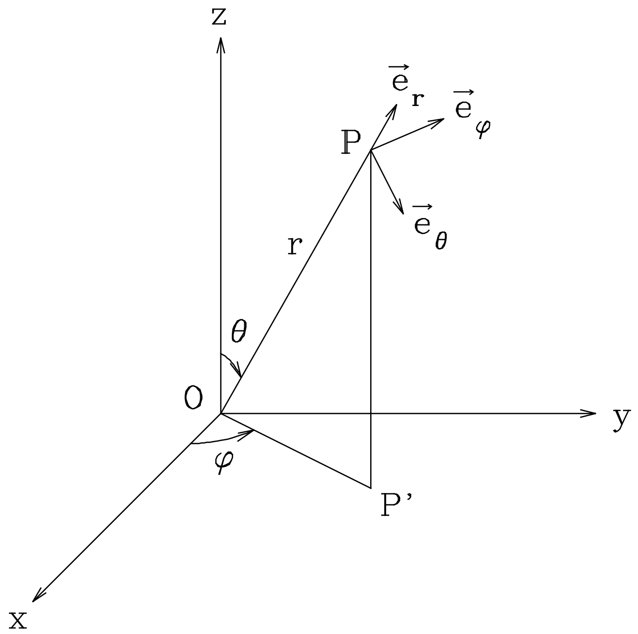

Figure 12.1: The spherical coordinates.

$$\begin{aligned}
\operatorname{curl}\vec{a} &= \left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(a_\phi\sin\theta\right) - \frac{1}{r\sin\theta}\frac{\partial a_\theta}{\partial\phi}\right]\vec{e}_r + \left[\frac{1}{r\sin\theta}\frac{\partial a_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}\left(ra_\phi\right)\right]\vec{e}_\theta \\
&+ \left[\frac{1}{r}\frac{\partial}{\partial r}\left(ra_\theta\right) - \frac{1}{r}\frac{\partial a_r}{\partial\theta}\right]\vec{e}_\phi\,, \\
\Delta\alpha &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\alpha}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\alpha}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\alpha}{\partial\phi^2}\,.
\end{aligned}$$

Let

$$L^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\,,$$

where we recognize the square angular momentum operator used in quantum theory. It is, in a way, the angular part of the laplacian.

$$\Delta\alpha = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\alpha}{\partial r}\right) - \frac{1}{r^2}L^2\alpha\,.$$

The spherical functions $Y_{lm}(\theta,\phi)$ are the eignefunctions of L^2:

$$L^2Y_{lm}(\theta,\phi) = \ell(\ell+1)Y_{lm}(\theta,\phi)\,.$$

12.2 Perturbation equations

Consider a perturbation with a e^{st} time dependence. We write the displacement vector as

$$\overrightarrow{\delta r} = \delta r\,\vec{e}_r + r\,\delta\theta\,\vec{e}_\theta + r\,\delta\phi\,\sin\theta\,\vec{e}_\phi\,.$$

Note that here δr is not the magnitude of $\overrightarrow{\delta r}$, but rather its radial component.

Using the eulerian perturbations, the differential equations of the problem can be written as follows.

Continuity equation:

$$\rho' + \delta r \frac{d\rho}{dr} + \rho \left\{ \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \delta r) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}(\sin\theta \, \delta\theta) + \frac{\partial \, \delta\phi}{\partial \phi} \right\} = 0 \, .$$

Momentum equations:

$$\begin{aligned} s^2 \delta r &= -\frac{\partial \Phi'}{\partial r} + \frac{\rho'}{\rho^2} \frac{dP}{dr} - \frac{1}{\rho} \frac{\partial P'}{\partial r} \, , \\ s^2 r \, \delta\theta &= -\frac{1}{r} \frac{\partial \Phi'}{\partial \theta} - \frac{1}{\rho r} \frac{\partial P'}{\partial \theta} \, , \\ s^2 r \sin\theta \, \delta\phi &= -\frac{1}{r \sin\theta} \frac{\partial \Phi'}{\partial \phi} - \frac{1}{\rho r \sin\theta} \frac{\partial P'}{\partial \phi} \, . \end{aligned}$$

Poisson equation:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi'}{\partial r} \right) - \frac{1}{r^2} L^2 \Phi' = 4\pi G \rho' \, .$$

Energy equation:

$$\begin{aligned} sT \left(S' + \delta r \frac{dS}{dr} \right) &= \epsilon' + \frac{\rho'}{\rho^2} \frac{1}{r^2} \frac{d}{dr}(r^2 F) \\ & -\frac{1}{\rho} \left\{ \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 F'_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta \, F'_\theta) + \frac{1}{r \sin\theta} \frac{\partial F'_\phi}{\partial \phi} \right\} \, . \end{aligned}$$

Transport equations:

$$\begin{aligned} F'_r &= -\lambda' \frac{dT}{dr} - \lambda \frac{\partial T'}{\partial r} \, , \\ F'_\theta &= -\frac{\lambda}{r} \frac{\partial T'}{\partial \theta} \, , \\ F'_\phi &= -\frac{\lambda}{r \sin\theta} \frac{\partial T'}{\partial \phi} \, . \end{aligned}$$

We can write P' and T' in terms of ρ', S' and the components of the displacement. There remains 9 unknowns: ρ', S', δr, $\delta\theta$, $\delta\phi$, F'_r, F'_θ, F'_ϕ and Φ', which must satisfy the 9 partial differential equations.

This problem is quite complicated. In particular, it is not easy to write the surface boundary conditions for the components of the flux. We will restrict ourselves to the study of the non-radial, adiabatic oscillations. The energy equation is replaced by $\delta S = 0$. It is no longer necessary to determine $\overrightarrow{F'}$ and the transport equations become unnecessary as well.

$\delta\theta$ and $\delta\phi$ can be determined from the momentum equations:

$$\begin{aligned} \delta\theta &= -\frac{1}{s^2 r^2}\frac{\partial\chi}{\partial\theta}\,, \\ \delta\phi &= -\frac{1}{s^2 r^2 \sin^2\theta}\frac{\partial\chi}{\partial\phi}\,, \end{aligned}$$

where we have defined for simplicity $\chi = \Phi' + P'/\rho$. We substitute these expressions in the continuity equation:

$$\rho' + \delta r\,\frac{d\rho}{dr} + \frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2\delta r) + \frac{\rho}{s^2 r^2}L^2\chi = 0\,.$$

We develop δr, Φ', ρ' and P' as series of spherical functions:

$$\delta r(r,\theta,\phi,t) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\delta r_{\ell m}(r)Y_{\ell m}(\theta,\phi)e^{st}\,,$$

$$\ldots$$

In these differential equations, the derivatives with respect to θ and ϕ only appear in the L^2 operator. As the spherical functions are eigenfunctions of this operator, the equations can be separated. We get ordinary differential equations for the radial functions $\delta r_{\ell m}(r)$, ... We therefore get, for each couple (ℓ, m), a fourth order differential system of the form (we omit the indices ℓ and m):

$$\begin{gathered} \rho' + \delta r\frac{d\rho}{dr} + \frac{\rho}{r^2}\frac{d}{dr}(r^2\delta r) + \frac{\rho\ell(\ell+1)}{s^2r^2}\left(\Phi' + \frac{P'}{\rho}\right) = 0\,, \\ s^2\delta r = -\frac{d\Phi'}{dr} + \frac{\rho'}{\rho^2}\frac{dP}{dr} - \frac{1}{\rho}\frac{dP'}{dr}\,, \\ \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi'}{dr}\right) - \frac{\ell(\ell+1)}{r^2}\Phi' = 4\pi G\rho'\,. \end{gathered}$$

We still have to specify the boundary conditions which must be satisfied by the solutions of this system. At $r = 0$, some coefficients of the differential system are singular. We'll impose that the solutions remain regular. A series development shows that we must impose two boundary conditions at the center, and that in its neighborhood we must have

$$\delta r \propto r^{\ell-1}\,, \quad P' \text{ and } \Phi' \propto r^{\ell}\,.$$

At the stellar surface, we'll impose that $\delta P = 0$. It is unnecessary to refine this condition when using the adiabatic approximation.We'll also impose the continuity of the gravity potential and of its gradient. To write it, remember that outside the star Φ'_e (we will use the index e to designate the exterior) satisfies the Laplace equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi'}{dr}\right) - \frac{\ell(\ell+1)\Phi'}{r^2} = 0\,,$$

whose regular solution (i.e. the one which goes to zero at infinity) can be written as

$$\Phi'_e = \frac{A}{r^{\ell+1}}$$

where A is a constant.

The continuity of the potential and its derivative with respect to r at the stellar surface can be written as:

$$\delta\Phi = \delta\Phi_e\,,$$
$$\delta\frac{d\Phi}{dr} = \delta\frac{d\Phi_e}{dr}\,,$$

or

$$\Phi' + \delta r\frac{d\Phi}{dr} = \Phi'_e + \delta r\frac{d\Phi_e}{dr}\,,$$
$$\frac{d\Phi'}{dr} + \delta r\frac{d^2\Phi}{dr^2} = \frac{d\Phi'_e}{dr} + \delta r\frac{d^2\Phi_e}{dr^2}\,.$$

We note that the first derivatives of the potential are equal

$$\frac{d\Phi}{dr} = \frac{d\Phi_e}{dr}\,.$$

The second derivatives are equal only if the density goes to zero at the surface. Indeed we have

$$\frac{d^2\Phi_e}{dr^2} + \frac{2}{r}\frac{d\Phi_e}{dr} = 0\,,$$
$$\frac{d^2\Phi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr} = 4\pi G\rho\,.$$

Substracting these equations we get

$$\frac{d^2\Phi_e}{dr^2} - \frac{d^2\Phi}{dr^2} = -4\pi G\rho\,.$$

The conditions of continuity then give

$$\Phi' = \frac{A}{r^{\ell+1}}\,,$$
$$\frac{d\Phi'}{dr} = -\frac{(\ell+1)A}{r^{\ell+2}} - 4\pi G\rho\,\delta r\,.$$

We eliminate the constant A to get the required condition

$$\frac{d\Phi'}{dr} + \frac{\ell+1}{r}\Phi' + 4\pi G\rho\,\delta r = 0\,.$$

We must now solve, for each value of (ℓ, m), a homogeneous system of ordinary differential equations with boundary conditions. For an arbitrary value of s, the only solution is zero. It is only for some particular values of s, called eigenvalues, that non-zero solutions will exist. For each value of (ℓ, m), there is an infinity of solutions, which we will write as $s_{k\ell m}$.

The m index does not appear in the differential equations, nor in the boundary conditions. Therefore, we have

$$s_{k\ell m} = s_{k\ell m'} \,.$$

The eigenvalues can therefore be written using two indices only $s_{k\ell}$. Each eigenvalue thus corresponds to $2\ell+1$ different values of m, and therefore to $2\ell+1$ different non radial oscillations. One says that it is $2\ell+1$ times degenerate. The eigenfunctions describing these $2\ell+1$ modes have the same radial factor, and differ only through their angular factor. This degeneracy is due to the spherical symmetry of the equilibrium configuration. It also appears in the theory of the hydrogen atom in quantum theory. It is possible to show the existence of this degeneracy through the theory of groups. This degeneracy can be lifted by something which breaks the spherical symmetry, such as rotation.

12.3 The Cowling approximation

In non radial oscillations, the perturbations of the gravific potential are generally rather small. This is especially true for the high order modes (k or ℓ high). If we neglect them, we get a second order system of differential equations, which is easier to solve. This is the Cowling approximation.

$$\rho' + \delta r \frac{d\rho}{dr} + \frac{\rho}{r^2}\frac{d}{dr}(r^2 \delta r) + \frac{\ell(\ell+1)P'}{s^2 r^2} = 0 \,,$$
$$s^2 \delta r = \frac{\rho'}{\rho^2}\frac{dP}{dr} - \frac{1}{\rho}\frac{dP'}{dr} \,.$$

We assume that Γ_1 is constant and make the following change of variables

$$v = r^2 \delta r \, P^{1/\Gamma_1} \,, \qquad w = P'/P^{1/\Gamma_1} \,.$$

We use $s = -i\sigma$, and after some calculations we get

$$\begin{aligned} \frac{dv}{dr} &= \Big(\frac{L_\ell^2}{\sigma^2} - 1\Big)\frac{r^2 P^{2/\Gamma_1}}{\rho c^2} w \,, \\ \frac{dw}{dr} &= (\sigma^2 - n^2)\frac{\rho}{r^2 P^{2/\Gamma_1}} v \,. \end{aligned}$$

At the boundaries $r=0$ and $r=R$ we impose $v=0$. The parameters L_ℓ and n have the dimensions of frequencies. They are the Lamb frequency and the Brunt-Väisälä frequency, respectively; they are defined by the relations

$$L_\ell^2 = \frac{\ell(\ell+1)c^2}{r^2} \qquad \text{and} \qquad n^2 = -Ag \,,$$

with

$$A = \frac{d\ln\rho}{dr} - \frac{1}{\Gamma_1}\frac{d\ln P}{dr} \,.$$

We recall that the criterion of stability towards convection (Schwarzschild criterion) is given by $A < 0$ or $n^2 > 0$.

If Γ_1 is not constant, we can get a similar system through a more complicated change of variables.

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The reader will find in Dupret (2001) and Dupret et al. (2002) a detailed treatment of the outer boundary conditions in the non radial non adiabatic case and a detailed treatment of the pulsation in the very external layers of the star.

It is possible to show through the theory of groups that the frequencies degeneracy results from the spherical symmetry of the unperturbed configuration (Perdang, 1968).

The change of variables which must be done in the case of the Cowling approximation when Γ_1 is not constant, is given by Gabriel and Scuflaire (1979).

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Chapter 13

Non radial modes

13.1 Orthogonality of the eigenfunctions

We have seen previously that the momentum equation of an adiabatic perturbation can be written as

$$\sigma^2 \overrightarrow{\delta r} = \mathcal{L}\, \overrightarrow{\delta r}\,,$$

where $\mathcal{L}$ is a self-adjoint operator relative to the scalar product

$$(\vec{u}, \vec{v}) = \int \rho \vec{u} \cdot \overline{\vec{v}}\, dV\,.$$

The properties we established previously are valid for non radial adiabatic oscillations. In particular the eigenfunctions of this problem can be chosen to be orthogonal. Note that the eigenfunctions corresponding to the same frequency but with different ℓ or m indices are orthogonal because of the orthogonality of the spherical functions $Y_{\ell m}$.

13.2 Components of the displacement

For a given mode we have

$$\begin{aligned}
\overrightarrow{\delta r} &= \delta r\, \vec{e}_r + r\, \delta\theta\, \vec{e}_\theta + r \sin\theta\, \delta\phi\, \vec{e}_\phi \\
&= \delta r\, \vec{e}_r + \frac{1}{r\sigma^2}\left(\frac{\partial \chi}{\partial \theta}\vec{e}_\theta + \frac{1}{\sin\theta}\frac{\partial \chi}{\partial \phi}\vec{e}_\phi\right)\,, \\
\delta r &= \delta r(r)\, Y_{\ell m}(\theta, \phi)\, e^{-i\sigma t}\,, \\
\chi &= \chi(r)\, Y_{\ell m}(\theta, \phi)\, e^{-i\sigma t}\,.
\end{aligned}$$

We write

$$\overrightarrow{\delta r} = (a(r)\vec{\epsilon} + b(r)\vec{\eta})\, e^{-i\sigma t}\,,$$

with

$$a(r) = \delta r(r)\,,$$

$$
\begin{aligned}
b(r) &= \frac{\chi(r)}{r\sigma^2}\,, \\
\vec{\epsilon} &= Y_{\ell m}(\theta,\phi)\,\vec{e}_r\,, \\
\vec{\eta} &= \frac{\partial Y_{\ell m}}{\partial\theta}\vec{e}_\theta + \frac{1}{\sin\theta}\frac{\partial Y_{\ell m}}{\partial\phi}\vec{e}_\phi = \frac{\partial Y_{\ell,m}}{\partial\theta}\vec{e}_\theta + \frac{imY_{\ell,m}}{\sin\theta}\vec{e}_\phi\,.
\end{aligned}
$$

We note that

$$
\begin{aligned}
\int |\epsilon|^2\,d\Omega &= 1\,, \\
\int |\eta|^2\,d\Omega &= \ell(\ell+1)\,.
\end{aligned}
$$

We justify the last equality as follows

$$
\begin{aligned}
\int |\eta|^2\,d\Omega &= \int_0^{2\pi} d\phi \int_0^{\pi} \Big[\Big|\frac{\partial Y_{\ell m}}{\partial\theta}\Big|^2 + \frac{m^2}{\sin^2\theta}\,|Y_{\ell m}|^2\Big]\sin\theta\,d\theta \\
&= \int_0^{2\pi} d\phi\Big\{\Big[\sin\theta\;\overline{Y}_{\ell m}\frac{\partial Y_{\ell m}}{\partial\theta}\Big]_0^{\pi} \\
&\qquad + \int_0^{\pi}\Big[-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial Y_{\ell m}}{\partial\theta}\Big) + \frac{m^2}{\sin^2\theta}Y_{\ell m}\Big]\overline{Y}_{\ell m}\sin\theta\,d\theta\Big\}\,.
\end{aligned}
$$

The integrated term gives zero and

$$
\Big[-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial Y_{\ell m}}{\partial\theta}\Big) + \frac{m^2}{\sin^2\theta}Y_{\ell m}\Big] = L^2 Y_{\ell m} = \ell(\ell+1)Y_{\ell m}\,.
$$

Therefore

$$
\int |\eta|^2 d\Omega = \ell(\ell+1)\int_0^{2\pi} d\phi \int_0^{\pi} |Y_{\ell m}|^2 \sin\theta\,d\theta = \ell(\ell+1)
$$

and finally

$$
\int |\overrightarrow{\delta r}|^2 dm = \int \rho r^2\Big[a^2 + \ell(\ell+1)b^2\Big]dr\,.
$$

13.3 p, g and f modes

The differential system describing the non radial oscillations of a star can only be solved analytically in the very unrealistic case of a homogeneous model. This case will nevertheless reveal the different types of non radial modes (see figure 13.1).

For each couple of indices (ℓ,m), we get an infinity of unstable modes ($\sigma^2 = -s^2 < 0$). The values of σ^2 present an accumulation point in 0. The buoyancy force plays a preponderant role in the dynamics of these modes. They are similar to the internal gravity modes. In fact they describe the convective instability of the homogeneous model. They are called g modes (g = gravity).

There is also an infinity of stable modes ($\sigma^2 = -s^2 > 0$). The values of σ^2 do not have an accumulation point at finite distance. The pressure forces play a dominant role in the dynamics of these modes. They are similar to acoustic modes. They are called p modes (p = pressure).

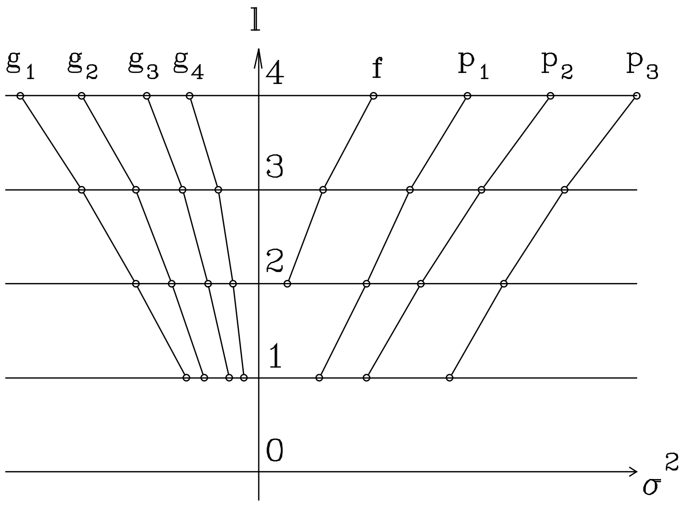

Figure 13.1: Frequencies of non radial modes of the homogeneous model.

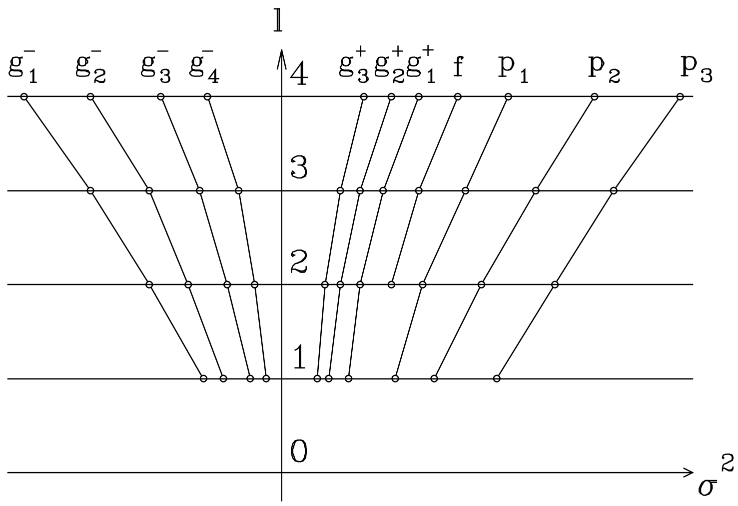

Figure 13.2: Frequencies of the non radial modes of a physical model.

Finally when $\ell > 1$, there is a stable mode, whose frequency is lower than those of the p modes. It is called the f mode (f = fundamental).

For more realistic models, the differential system is too complicated and cannot be solved analytically. Numerous numerical integrations show that the non radial modes of physically realistic models can be classified as those of the homogeneous model. The result is exactly the same if the model is entirely convective. If the model is entirely radiative, the g modes are stable. Their frequencies are lower than the f mode and p modes frequencies and have an accumulation point in 0. If the model has both radiative and convective zones, there are two spectra of g modes, one stable, the other unstable, as shown in figure 13.2. The stable g modes are labelled g^+ and the unstable g modes are labelled g^-.

In the Cowling approximation it is possible to rigourously demonstrate the existence of these different types of modes. Less rigorously, we can see that for high values of σ^2, neglecting the $1/\sigma^2$ term, we have a Sturm-Liouville problem with $\lambda = \sigma^2$ as parameter.

Table 13.1: Some characteristics of non radial modes of degree $\ell = 2$ for the standard model.

mode	ω	ξ_s/ξ_c	$\langle x\rangle$
g_{10}	0.567887	3.977(−3)	0.299
...	...	...	...
g_3	1.34992	−2.518(−2)	0.280
g_2	1.68171	5.521(−2)	0.278
g_1	2.21688	−0.2399	0.292
f	2.85926	3.6763	0.493
p_1	3.90687	−57.34	0.702
p_2	5.16947	213.0	0.735
p_3	6.43999	−453.2	0.742
...	...	...	...
p_{10}	15.2849	4204	0.739

The corresponding solutions are the p modes. For small values of σ^2, neglecting the σ^2 term, we have a Sturm-Liouville problem with $\lambda = 1/\sigma^2$ as parameter. The corresponding solutions are the g modes.

The non radial modes can be physically described as follows. The p modes are acoustic modes. The g^- modes describe the convective instability. The g^+ modes are internal gravity waves. In very concentrated models, the low k order p and g^+ modes can present a mixed character and behave as gravity waves in the central regions of the star and as acoustic waves in the external layers.

Table 13.1 gives ω, ξ_s/ξ_c and $\langle x\rangle$ for a few non radial modes of the standard model (polytrope of index 3 with $\Gamma_1 = 5/3$), with

$$\xi = x^{1-\ell}\frac{\delta r}{R} \qquad \text{and} \qquad \langle X\rangle = \frac{\displaystyle\int X|\overrightarrow{\delta r}|^2 dm}{\displaystyle\int |\overrightarrow{\delta r}|^2 dm}.$$

The figures 13.3 to 13.7 show ξ for a few non radial modes of the standard model. We note that for the p modes, as for the radial modes, ξ grows in the external layers of the star. On the contrary, for the g modes, ξ is larger in the central regions.

13.4 Spheroidal and toroidal modes

Do the radial and non radial modes studied so far form a complete set ? We will show that they do not, and how it can be completed.

Any vector field can be written as

$$\vec{u} = \operatorname{grad}\phi + \operatorname{curl}\vec{v}\,.$$

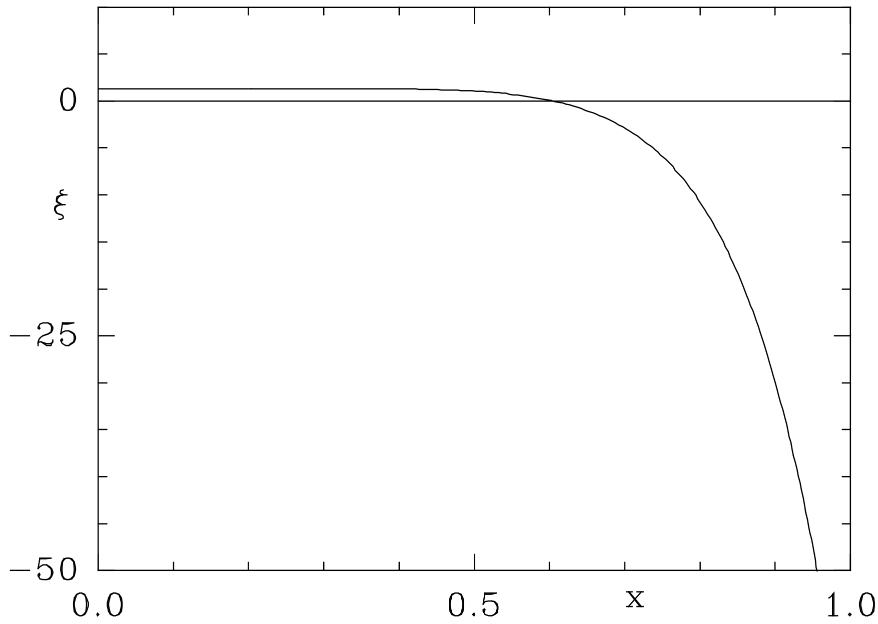

Figure 13.3: Standard model, non radial oscillation mode $\ell = 2,\ p_1$.

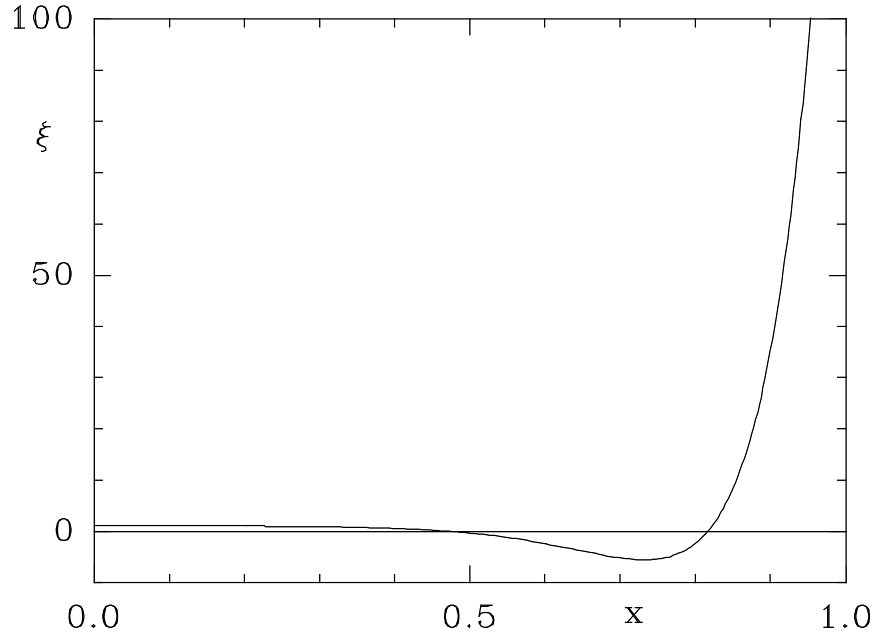

Figure 13.4: Standard model, non radial oscillation mode $\ell = 2,\ p_2$.

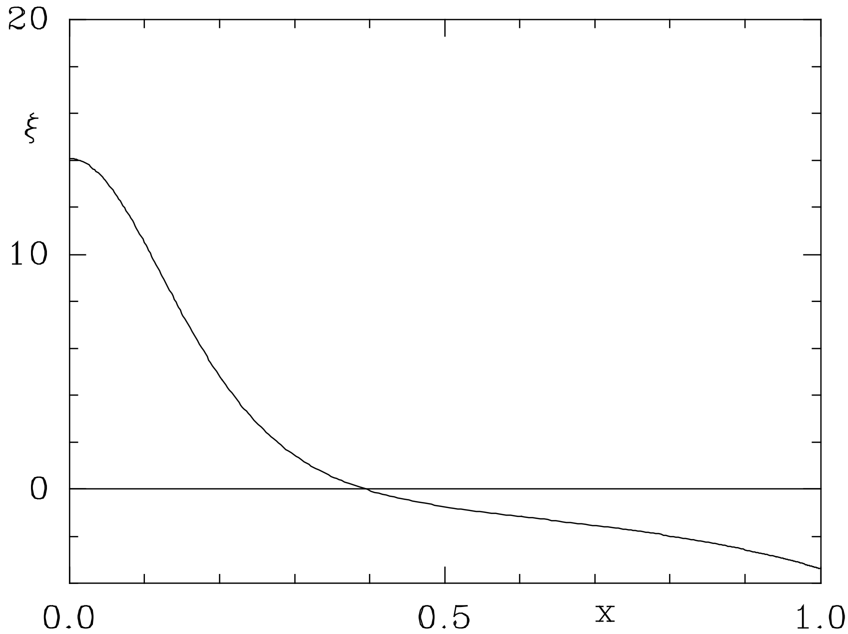

Figure 13.5: Standard model, non radial oscillation mode $\ell = 2$, g_1.

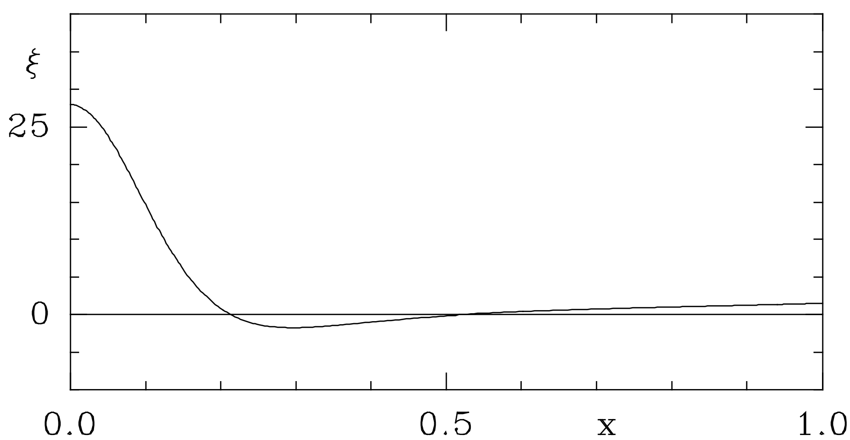

Figure 13.6: Standard model, non radial oscillation mode $\ell = 2$, g_2.

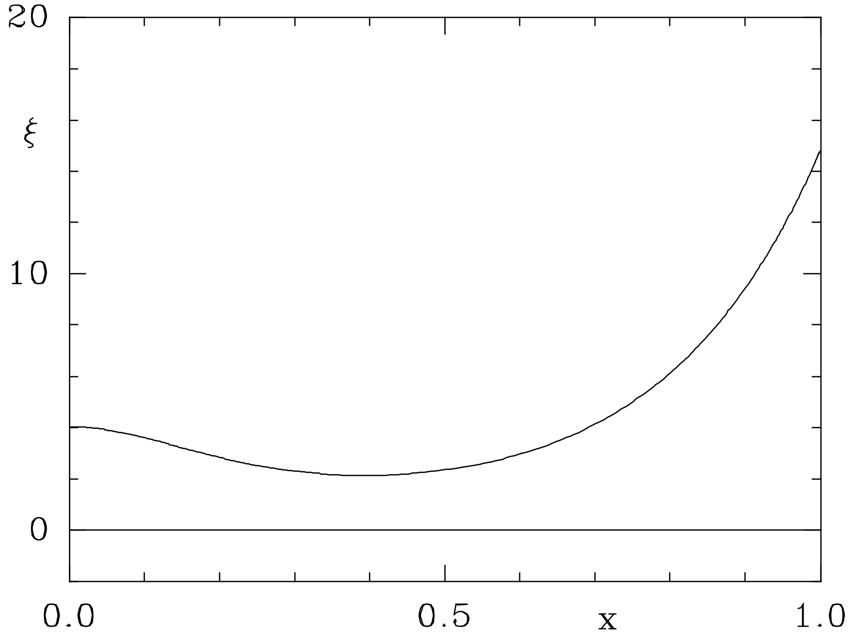

Figure 13.7: Standard model, non radial oscillation mode $\ell = 2$, f.

Under some conditions (behavior of the fields at infinity), this decomposition is unique. The $\mathrm{grad}\,\phi$ term is called the longitudinal component of the field and $\mathrm{curl}\,\vec{v}$ is its transversal component. The transversal component (also called solenoidal component) can be decomposed as

$$\mathrm{curl}\,\vec{v} = \mathrm{curl}(\chi\vec{e}_r) + \mathrm{curl}\,\mathrm{curl}(\psi\vec{e}_r)\,.$$

The $\mathrm{curl}(\chi\vec{e}_r)$ term is a toroidal field and $\mathrm{curl}\,\mathrm{curl}(\psi\vec{e}_r)$ is a poloidal field.

A vector field $\vec{u}$ can thus be described by three scalar potentials ϕ, χ and ψ.

$$\vec{u} = \mathrm{grad}\,\phi + \mathrm{curl}(\chi\vec{e}_r) + \mathrm{curl}\,\mathrm{curl}(\psi\vec{e}_r)\,.$$

The poloidal term can be developed, and we get

$$\vec{u} = \alpha\vec{e}_r + \mathrm{grad}\,\beta + \mathrm{curl}(\chi\vec{e}_r)\,.$$

We note that in these two expressions the toroidal term is uniquely determined by the vector field $\vec{u}$.

We now write $\overrightarrow{\delta r}$ using the above expressions. The momentum equation can be written as

$$s^2\overrightarrow{\delta r} = -\,\mathrm{grad}\,\Phi' - \frac{1}{\rho}\,\mathrm{grad}\,P' + \frac{\rho'}{\rho^2}\,\mathrm{grad}\,P\,.$$

Using the adiabatic relation and the continuity equation we get

$$s^2\overrightarrow{\delta r} = -\,\mathrm{grad}\,\chi + c^2\vec{A}\,\mathrm{div}\,\overrightarrow{\delta r}\,,$$

where

$$\begin{aligned}
\chi &= \phi' + \frac{P'}{\rho}\,, \\
\vec{A} &= \frac{1}{\rho}\,\mathrm{grad}\,\rho - \frac{1}{\Gamma_1 P}\,\mathrm{grad}\,P\,.
\end{aligned}$$

We have thus written $\overrightarrow{\delta r}$ as

$$\overrightarrow{\delta r} = \alpha \vec{e}_r + \mathrm{grad}\, \beta \,,$$

with

$$\begin{aligned} \alpha &= c^2 A \,\mathrm{div}\, \overrightarrow{\delta r}/s^2 \,, \\ \beta &= -\chi/s^2 \,. \end{aligned}$$

The non radial modes we have studied so far do not have any toroidal component. It is therefore obvious that they do not form a complete set. To obtain a complete set we must also consider the 0 frequency modes, which were so far neglected. We will not develop this point in details.

The 0 frequency modes are divergenceless. We will distinguish two classes.

1. There are three spheroidal modes $\ell = 1$, $m = -1, 0, 1$. The radial component is a non-zero constant. They describe a solid translation of the star.

   $$\overrightarrow{\delta r} = a \Big\{ Y_{\ell m} \vec{e}_r + \frac{\partial Y_{\ell m}}{\partial \theta} \vec{e}_\theta + \frac{1}{\sin\theta} \frac{\partial Y_{\ell m}}{\partial \phi} \vec{e}_\phi \Big\} = \text{constant vector.}$$

   Formally we could consider them as f modes of degree $\ell = 1$.

2. There is an infinity of toroidal modes. They are characterized by the absence of radial displacement. They are of the form

   $$\overrightarrow{\delta r} = a(r) \Big\{ \frac{1}{\sin\theta} \frac{\partial Y_{\ell m}}{\partial \phi} \vec{e}_\theta - \frac{\partial Y_{\ell m}}{\partial \theta} \vec{e}_\phi \Big\} \,.$$

   The horizontal and divergenceless displacements do not of course perturb the hydrostatic equilibrium of the star and we have

   $$\rho' = 0, \qquad P' = 0, \qquad \Phi' = 0 \,.$$

The eigenfunctions $\overrightarrow{\delta r}$ of the radial and of the non radial problems (zero and non zero frequency modes) form a complete set. Any displacement field $\overrightarrow{\delta r}$ can be written as a series in terms of elements of this set.

13.5 Asymptotic expression for the frequencies

The study of the asymptotic behavior of the non radial modes is more complicated than that of the radial modes, even in the Cowling approximation. From the equations for v and w obtained previously we can write a second order equation in v or in w. In addition to the singularities at the center and at the surface there is also a moving singularity at the position where $\sigma^2 = \sigma_a^2$ in the first case, and one or several moving singularities where $\sigma^2 = n^2$ in the second case. As for the asymptotic study of the radial oscillations, the model is divided into several zones, each with one single singularity. The approximate solutions obtained in each zone are then joined continuously. We will give without proof the lowest order approximation for the frequencies.

For the p modes we have

$$\sigma_{kl} \approx \frac{\left(k + \frac{\ell}{2} + \frac{n_e}{2} + \frac{1}{4}\right)\pi}{\int_0^R \frac{dr}{c}}.$$

For $\ell = 0$, this expression reduces to the one obtained above for radial modes, numbered as above. We note the frequency equidistance and approximate superpositions given by

$$\sigma_{k+1,l} - \sigma_{k,l} \approx \text{const}\,, \qquad \sigma_{kl} \approx \sigma_{k-1,l+2} \qquad \text{and} \qquad \sigma_{k,l+1} \approx (\sigma_{k,l} + \sigma_{k+1,l})/2.$$

For the $g^{\pm}$ modes, the asymptotic frequencies are given by

$$\frac{\sqrt{\ell(\ell+1)}}{|\sigma_{kl}|} \approx \frac{\left(k + \frac{\ell}{2} + C\right)\pi}{\int \frac{|n|}{r} dr}.$$

Here, the integral is computed over the radiative zone for a g^+ mode, and over the convective zone for a g^- mode. The constant C depends on the position of the integration zone in the model (central regions or superficial layer). It is even more complicated if there are several radiative or convective zones.

References

The paper by Smeyers (1984) on non radial modes is particularly interesting.

For more information on the physical meaning of the different types of modes, we suggest the papers by Scuflaire (1974ab). Tolstoy (1963) gives a remarquable account of the theory of plane waves in a simple geometric context.

For a rigorous study of the non radial oscillations in the Cowling approximation, with variable Γ_1, we recommend the paper by Gabriel and Scuflaire (1979). It also contains the proof of existence of the p, $g^{\pm}$ and f modes. Christensen-Dalsgaard and Gough (2001) have developed an interesting reflexion about the classification of non radial modes and the status of the $\ell = 1$ f mode.

We find a detailed discussion of the decomposition of $\overrightarrow{\delta r}$ in its spheroidal and toroidal components in the paper by Aizenman and Smeyers (1977).

We recommend the papers by Kaniel and Kovetz (1967) and Eisenfeld (1969) for further information about the completeness of the eigenfunctions of the non radial problem.

The asymptotic behavior of the non radial modes in the Cowling approximation is very well described in Tassoul (1980). There are numerous references and comments on previous studies in this paper. In later papers, it became possible to go further than the

Cowling approximation: Tassoul (1990) and Smeyers et al. (1996) for p modes ; Smeyers et al. (1995) and Willems et al. (1997) for g modes.

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Chapter 14

Influence of rotation

We study the effect of a slow rotation of the star on the non radial adiabatic pulsations. The model undergoes differential rotation around the z-axis at angular velocity $\Omega(r,\theta)$ and the rotation is sufficiently slow to ignore the Ω^2 terms (if they had to be accounted for, the model could no longer be considered spherical).

Compared to the case without rotation, the difference appears in the development of the momentum equation

$$\frac{d^2\overrightarrow{\delta r}}{dt^2} = \mathcal{L}\,\overrightarrow{\delta r},$$

which now becomes

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot \mathrm{grad}\right)^2 \overrightarrow{\delta r} = \mathcal{L}\,\overrightarrow{\delta r},$$

where $\vec{v}$ is the rotation velocity in the unperturbed model,

$$\vec{v} = \Omega r \sin\theta \overrightarrow{e_\phi}.$$

We look for a solution with a time-dependence $e^{-i\sigma t}$ and we neglect the second-order terms in v (or in Ω), and we get

$$\sigma^2\overrightarrow{\delta r} + 2\sigma\mathcal{M}\,\overrightarrow{\delta r} + \mathcal{L}\,\overrightarrow{\delta r} = 0,$$

where

$$\mathcal{M}\,\overrightarrow{\delta r} = i(\vec{v}\cdot \mathrm{grad})\,\overrightarrow{\delta r}.$$

$\mathcal{M}$ is a purely imaginary linear operator ($\overline{\mathcal{M}} = -\mathcal{M}$) and it is easy to show that it is hermitian. Indeed

$$\begin{aligned}(\mathcal{M}\vec{\xi},\vec{\eta}) &= \int i\rho\left[(\vec{v}\cdot \mathrm{grad})\vec{\xi}\right]\cdot\overline{\vec{\eta}}\,dV \\ &= \int i\,\mathrm{div}\left[\rho(\vec{\xi}\cdot\overline{\vec{\eta}})\vec{v}\right]\,dV - \int i(\vec{\xi}\cdot\overline{\vec{\eta}})\,\mathrm{div}(\rho\vec{v})\,dV - \int i\rho\left[(\vec{v}\cdot \mathrm{grad})\overline{\vec{\eta}}\right]\cdot\vec{\xi}\,dV.\end{aligned}$$

The first term can be transformed into a surface integral which vanishes, and the second one contains the zero term $\mathrm{div}(\rho\vec{v})$. We then obtain $(\vec{\xi},\mathcal{M}\vec{\eta})$.

For any function $\vec{\xi}$, it is possible to write

$$J = (\xi,\xi)\,, \qquad M = (\xi,\mathcal{M}\xi) \qquad \text{and} \qquad L = (\xi,\mathcal{L}\xi)$$

and to solve the equation

$$J\Sigma^2 + 2M\Sigma + L = 0$$

with respect to Σ. We thus define a functional $\Sigma(\xi)$ which has an interesting variational property. If ξ is a solution of the momentum equation for a real value of σ then ξ makes the functional Σ stationary and $\sigma = \Sigma(\xi)$.

We now look on an oscillation mode. In the absence of rotation it is described by an eigenfunction ξ_0 and its frequency σ_0 obeys the equation

$$J_0\sigma_0^2 + L_0 = 0 \qquad \text{with} \qquad J_0 = (\xi_0, \xi_0) \qquad \text{and} \qquad L_0 = (\xi_0, \mathcal{L}\xi_0).$$

In the presence of slow rotation its eigenfunction and its eigenfrequency are written as

$$\xi = \xi_0 + \xi_1 \qquad \text{and} \qquad \sigma = \sigma_0 + \sigma_1,$$

where ξ_1 and σ_1 are small corrections due to the rotation. The frequency obeys the equation

$$J\sigma^2 + 2M\sigma + L = 0.$$

We develop this equation neglecting the terms higher than first order terms in the correction, noting that $M_0 = (\xi_0, \mathcal{M}\xi_0)$ must also be considered as a correction since this expression contains the rotation velocity of the star. We get

$$\sigma_1 = -M_0/J_0.$$

It is easy to show that

$$\mathcal{M}\vec{\xi} = -m\Omega\vec{\xi} + i\vec{\Omega} \times \vec{\xi},$$

and we get

$$\sigma_1 = \frac{\displaystyle\int \rho \left[m\Omega|\xi|^2 - i(\vec{\Omega}, \vec{\xi}, \bar{\vec{\xi}}) \right] dV}{\displaystyle\int \rho|\xi|^2 dV}.$$

This expression is zero for a radial mode. For a non radial mode $\vec{\xi}$ can be written as

$$\vec{\xi} = aY_{\ell m}\vec{e}_r + b\left(\frac{\partial Y_{\ell m}}{\partial\theta}\vec{e}_\theta + \frac{im}{\sin\theta}Y_{\ell m}\vec{e}_\phi\right)$$

and we get

$$i(\vec{\Omega}, \vec{\xi}, \bar{\vec{\xi}}) = 2m\Omega ab|Y_{\ell m}|^2 + m\Omega b^2 \frac{\partial |Y_{\ell m}|^2}{\partial\theta} \operatorname{cotg}\theta.$$

Finally,

$$\sigma_1 = \frac{m\displaystyle\int \rho\Omega \left\{ (a^2 - 2ab)|Y_{\ell m}|^2 + b^2 \left(\left|\frac{\partial Y_{\ell m}}{\partial\theta}\right|^2 + \frac{m^2}{\sin^2\theta}|Y_{\ell m}|^2 - \frac{\partial|Y_{\ell m}|^2}{\partial\theta} \operatorname{cotg}\theta \right) \right\} dV}{\displaystyle\int \rho r^2 [a^2 + \ell(\ell+1)b^2]\, dr}.$$

This expression can be simplified if we assume that $\Omega = \Omega(r)$. We then get

$$\sigma_1 = \frac{m \int \rho r^2 \Omega [a^2 + \ell(\ell+1) b^2 - 2ab - b^2] \, dr}{\int \rho r^2 [a^2 + \ell(\ell+1) b^2] \, dr}.$$

In the particular case of uniform rotation, this expression can be further simplified to get

$$\sigma_1 = m\beta\Omega \qquad \text{or} \qquad \sigma_{k\ell m} = \sigma^0_{k\ell} + m\beta_{k\ell}\Omega,$$

where $\sigma^0_{k\ell}$ is the eigenfrequency in the absence of rotation and the constant $\beta_{k\ell}$ is calculated from the eigenfunctions of the mode (k, ℓ) without rotation,

$$\beta_{k\ell} = \frac{\int \rho r^2 [a^2 + \ell(\ell+1) b^2 - 2ab - b^2] \, dr}{\int \rho r^2 [a^2 + \ell(\ell+1) b^2] \, dr}.$$

The rotation removes completely the degeneracy.

The rotation of the star also affects the toroidal modes. In the presence of rotation these modes acquire non zero frequencies and the displacement ceases to be purely horizontal and toroidal. These modes have low frequency and are similar to the Rossby or planetary waves. Their dynamics is dominated by the Coriolis force. We will not study them here. Note that in the case of uniform rotation, their frequencies are given, as first order approximation, by the relation

$$\sigma = m\Omega - \frac{2m\Omega}{\ell(\ell+1)}\,.$$

14.1 Non radial oscillations in variable stars

The evidence for non radial modes in variable stars lies on several observations: characteristic deformation of spectral lines, pulsation frequency lower than that of the fundamental radial mode, frequency ratios incompatible with radial pulsations, frequency multiplets produced by the frequency splitting due to the rotation.

Some β Cep variables and a number of δ Sct variables exhibit non radial pulsations in addition to their radial modes.

The variability of the spectral lines in some variable B stars (SPB or *slowly pulsating B stars*) comes from the existence of non radial modes in the presence of rotation.

Variable white dwarfs are usually multiperiodic and their periods go from a few hundreds to many thousands of seconds. The observed modes are g modes of low ℓ degree. Frequency splitting due to rotation is sometimes observed.

The variability of Ap stars is due to surface inhomogeneities and to rotation. In some of these stars, small amplitude light variations with periods between 4 and 15 minutes are superimposed to those. Their study yields regularly spaced frequencies, illustrating the rotational splitting.

The best studied case is the 5 minutes solar oscillation. It is made up of thousands of modes of all values of the ℓ index of the spherical functions between 0 and 3000, whose frequencies are around 3 mHz. For low values of ℓ, these are p modes of order between 10 and 30. For a given value of ℓ the frequency spacing $\Delta\nu = \Delta\sigma/2\pi$ of two consecutive modes is close to 136 μHz. The relative error on the frequencies is smaller than 10^{-4} for most of these modes and of the order 10^{-5} for some of them, so that the fine structures due to the rotation can be put forward (the rotational splitting in a multiplet is of the order of 0.4 μHz).

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Chapter 15

Helio- and asteroseismology

The principle goal of helioseismology or asteroseismology is to infer solar or stellar structure from the observed properties of the solar or stellar oscillations. This is often called the inverse problem. Much more observational data is available for the Sun than for other stars. We will therefore mostly look at helioseismology.

The simplest inversion method consists in having a family of models with few parameters and in adjusting their values, for example using the least mean square method, to best reproduce the observed frequencies. It is thus possible to adjust the helium abundance and the mixing length parameter in the convective zone. This method does not use sophisticated calculation techniques but it also does not extract a maximum of information from the available helioseismic data.

Another method uses the asymptotic expressions of the oscillation frequencies. These expressions have the advantage of putting forward the most important factors determining the frequencies, but their validity for not too high order modes is doubtful.

We will describe below some aspects of the numerical inversion methods. The first step is always to solve the direct problem, which consists in calculating the oscillation frequencies of a reference solar model. As we have seen, this means solving a system of differential eigenvalue equations with boundary conditions whose coefficients depend on the equilibrium model. The second step consists in getting corrections to those coefficients using the observed solar values. Of course helioseismology cannot provide a unique solution for the solar model. There certainly are solar properties that cannot be deduced from the oscillation frequencies, and in addition there is a finite amount of data, with observational errors.

15.1 Determination of $\Omega(r)$

The rotational splitting gives information on the rotation angular velocity Ω inside the Sun. For simplicity we will assume that it only depends on r.

$$\sigma_{k\ell m} - \sigma_{k\ell 0} = m \int K_{k\ell}(r)\Omega(r)\, dr \,,$$

where the kernel $K_{k\ell}(r)$ is constructed from the eigenfunctions of the (k,ℓ) mode in the absence of rotation. Using a unique index i for the couple (k,ℓ), the seismic data impose linear conditions on Ω

$$\int K_i(r)\Omega(r)\,dr = w_i\,,\quad i = 1,\ldots,N.$$

It is clear that this finite number of equations does not uniquely determine $\Omega(r)$. To each solution of these equations one can add a function $\Omega_\perp(r)$ orthogonal to all the kernels $K_i(r)$ and obtain a new solution.

$$\int K_i(r)\Omega_\perp(r)\,dr = 0\,,\quad i = 1,\ldots,N.$$

In addition there are errors on the data w_i, which reinforces the uncertainty on $\Omega(r)$. The problem is therefore not only to find an approximate solution of the equations above, but to find, amongst an infinity of solutions, the one that would best describe the real angular velocity distribution inside the Sun. In order to do that we would need more information on $\Omega(r)$, coming from a non-helioseismic source. Without such information, it is necessary to impose arbitrary conditions on $\Omega(r)$. We will outline two inversion methods.

Spectral development

Since the component of $\Omega(r)$ orthogonal to the $K_i(r)$ is not accessible to observations, it seems natural to determine only the part of $\Omega(r)$ which can be written as a linear combination of the $K_i(r)$:

$$\widetilde{\Omega}(r) = \sum_{j=1}^{N} \Omega_j K_j(r)\,.$$

The coefficients Ω_j must satisfy the following equations

$$\sum_{j=1}^{N} A_{ij}\Omega_j = w_i\,,\quad i = 1,\ldots,N \qquad \text{with} \qquad A_{ij} = A_{ji} = \int K_i(r)K_j(r)\,dr\,.$$

The problem is that the matrix A is almost singular. Small errors on the w_i lead to large errors on the Ω_j. The solution that would be obtained through the direct solving of this system of equations would be dominated by the errors on the data and would be totally wrong. This is called an ill-posed problem.

To show where the problem lies, we write the symmetric positive-definite matrix A as

$$A = U\,\mathrm{diag}(\lambda_1,\ldots,\lambda_N)\,\widetilde{U}\,,$$

where U is orthogonal and the eigenvalues are ordered by decreasing values $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0$ with $\lambda_N \ll \lambda_1$. If Ω is the vector of components Ω_j, we get

$$\Omega = U\mathrm{diag}\Big(\frac{1}{\lambda_1},\ldots\frac{1}{\lambda_N}\Big)\,\widetilde{U}w\,.$$

We see that the amplification of the data errors comes from the small eigenvalues of the matrix A. The reason is that the number of independent data is much smaller than the number of measured frequencies. In this case, the *singular value decomposition* technique can bring some help to obtain a *reasonable* solution.

Least square fit method

We choose a basis of functions $\phi_j(r)$ ($j = 1, \ldots, M$) to write an approximation $\widetilde{\Omega}(r)$ of Ω:

$$\widetilde{\Omega} = \sum_{j=1}^{M} \Omega_j \phi_j(r) \, .$$

Given the redundance of the data compared to the information they provide we choose $M < N$ and we determine the Ω_j using the least square fit method. We generally impose additional conditions on $\widetilde{\Omega}(r)$, e.g., that it does not vary too fast. We must then usually minimize an expression such as

$$S = \sum_{i=1}^{N} \left[w_i - \int K_i(r) \widetilde{\Omega}(r) \, dr \right]^2 + \mu \int \left(\frac{d^2 \widetilde{\Omega}}{dr^2} \right)^2 dr \, ,$$

where μ is an arbitrarily chosen positive parameter. The last term softens the variations of $\widetilde{\Omega}(r)$, but other expressions can be used. We then determine the Ω_j which minimize the value of S by solving the linear equations

$$\frac{\partial S}{\partial \Omega_j} = 0 \, , \quad j = 1, \ldots, M \, .$$

Results

The superficial rotation of the Sun has been known for a long time: the equatorial regions rotate faster than the polar regions. We cannot however get an agreement to better than 2% between the different observations. Thanks to helioseismology, it has been possible to obtain the rotational angular velocity as a function of depth and latitude with a precision of a few percents in the convective zone and the upper part of the radiative core (down to 0.4 $R_\odot$). In the convective zone, the rotation seems to be very similar to what it is at the surface, faster at the equator than in the polar regions. Under the convective zone, that rotation appears to be uniform (as in a solid) with an angular velocity intermediate between the equatorial and the polar values. We do not know much about the rotation below 0.4 $R_\odot$ because of the uncertainty on the splitting data for low ℓ modes. It seems however that the most internal layers rotate faster.

15.2 Determination of the solar structure

The adiabatic pulsation frequencies are determined by the distribution of mass and of Γ_1 in the model, i.e. by the functions $\rho(r)$ and $\Gamma_1(r)$. It is easy to see that the other coefficients in the pulsation equations can be deduced from these two functions. We could choose two other independent functions instead $\rho(r)$ and $\Gamma_1(r)$, such as for example $\rho(r)$ and $c(r)$ (the sound speed). The asymptotic expression of the p modes clearly shows the essential role played by the sound speed in the determination of the frequencies of these modes. In what follows, we will show how the seismic data give information on the sound speed. The process can be generalized to get simultaneously information on $c(r)$ and $\rho(r)$.

The pulsation frequencies depend on $c(r)$ in a complicated non linear way. The process used to determine $\Omega(r)$ can only be applied after linearization of the problem around a reference model. We examine how the frequency of a given mode (for simplicity we will omit the indices k, ℓ) is changed by a small change in the sound speed $\delta c(r)$. We will calculate $\delta\sigma$ neglecting the terms higher than first order in $\delta c(r)$. From

$$\sigma^2 \xi = -\mathcal{L}\xi \, ,$$

we easily get

$$\frac{\delta\sigma}{\sigma} = -\frac{(\xi, \delta\mathcal{L}\,\xi)}{2\sigma^2(\xi,\xi)} \, ,$$

where $\delta\mathcal{L}$ is the correction to the operator $\mathcal{L}$, linear in $\delta c(r)$. It is easy to get

$$\begin{aligned}
\operatorname{div} \overrightarrow{\xi} &= \left\{ \frac{1}{r^2}\frac{d}{dr}\left[r^2 a(r)\right] - \frac{\ell(\ell+1)}{r} b(r) \right\} Y_{\ell m}(\theta,\phi) \, , \\
(\xi, \delta\mathcal{L}\,\xi) &= -2\int \rho c^2 \left|\operatorname{div} \overrightarrow{\xi}\right|^2 \frac{\delta c}{c} \, dV \\
&= -2\int \rho c^2 \left\{ \frac{1}{r}\frac{d}{dr}(r^2 a) - \ell(\ell+1) b \right\}^2 \frac{\delta c}{c} \, dr \, , \\
(\xi,\xi) &= \int \rho r^2 \left[a^2 + \ell(\ell+1) b^2\right] \, dr \, .
\end{aligned}$$

We therefore have

$$\frac{\delta\sigma}{\sigma} = \int K(r) \frac{\delta c}{c} \, dr \, ,$$

with

$$K(r) = \frac{\rho c^2 \left\{ \frac{1}{r}\frac{d}{dr}(r^2 a) - \ell(\ell+1) b \right\}^2}{\sigma^2 \int \rho r^2 \left[a^2 + \ell(\ell+1) b^2\right] \, dr} \, .$$

Finally we must solve the system

$$\int K_i(r) \frac{\delta c}{c} \, dr = \frac{\sigma_{i\,obs} - \sigma_{i\,calc}}{\sigma_{i\,calc}} \equiv w_i \, , \qquad i = 1, \ldots, N \, .$$

We can use the linear inversion techniques mentionned earlier. If necessary, several correction cycles can be done.

The thickness of the convective zone and the sound speed inside the Sun have been obtained by the analysis of helioseismic data.

For stars other than the Sun, it has been possible to use seismic data from rapidly oscillating Ap stars, δ Sct variables and variable white dwarfs.

15.3 Non adiabatic asteroseismology

The techniques described above rely on the comparison between theoretical and observed frequencies and are almost independent of non adiabatic effects. But multicolor photometry, through amplitude ratios and phase differences, is able to bring to the fore the strongly non adiabatic behavior of the pulsation in the atmosphere of the star. The fitting of the theoretical predictions to this type of observation produces constraints on the structure of the outer layers of the star (convective zone, metallicity). The expression *non adiabatic asteroseismology* has been coined to designate this type of investigation.

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The singular value decomposition technique can also be applied to non square matrices. Examples can be found in the papers by Korzennik and Ulrich (1989), Christensen-Dalsgaard et al. (1990), Gough and Thompson (1991), Gu (1993) and Christensen-Dalsgaard and Thompson (1993).

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