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Convection-pulsation coupling⋆

I. A mixing-length perturbative theory

A. Grigahcène[1], M.-A. Dupret[1], M. Gabriel[2], R. Garrido[1], and R. Scuflaire[2]

[1] Instituto de Astrofísica de Andalucía-CSIC, Apartado 3004, 18080 Granada, Spain
[2] Institut d'Astrophysique et de Géophysique de l'Université de Liège, Belgium

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Abstract. We present in details a time-dependent convection treatment in the frame of the Mixing-Length Theory (MLT). Following the original ideas by Unno (1967, PASJ, 19, 140), this theory has been developed by Gabriel et al. (1974, Bull. Ac. Roy. Belgique, Classe des Sciences, 60, 866) and Gabriel (1996, Bull. Astron. Soc. India, 24, 233). In this paper, we present it in a united form, we detail the basic derivations and approximations and give final improvements. A new perturbation of the energy closure equation is proposed for the first time, making it possible to avoid the occurrence of short wavelength spatial oscillations of the thermal eigenfunctions. This theory accounts for the perturbation of the convective flux, the turbulent Reynolds stress and the turbulent kinetic energy dissipation. It has been numerically implemented in a non-radial non-adiabatic pulsation code and the first results published in a Letter by Dupret et al. (2004a, A&A, 414, L17) indicate that the theory predicts the observed red border of the lower end of the instability strip and the driving mechanism of the recently discovered γ Dor stars.

Key words. stars: oscillations – convection – stars: interiors

1. Introduction

Usually stellar oscillations are calculated starting from the assumption that they are perturbations of a static model where the gas is at rest. However, in convection zones where the gas is moving up and down, this assumption is no longer valid. At the beginning of any convection-pulsation interaction study, it thus seems necessary to make a clear distinction between the convective motions and the oscillations. This is generally done by introducing a more or less arbitrary cut-off in the frequency – wave number space of turbulence. In practice, it is supposed that the most energetic convective motions have short wavelengths and interact with the oscillations of longer wavelength. This distinction allows us to consider, on the one hand, the convective *fluctuations* corresponding to the difference between the physical conditions in a convective cell and in the average medium, and on the other hand, the *perturbation* of the mean structure corresponding to the oscillations. In this perturbative approach, the solution of the general equations for the unperturbed model is constrained in such a way that it does not contain the oscillations we want to study. This provides an unperturbed model. Then we study the stability of this solution when the constraints are relaxed. More precisely, the perturbation of the mean structure gives the pulsation equations, where convection-pulsation coupling terms (perturbation of the convective flux, turbulent pressure, ...) appear. These coupling terms are obtained by perturbing the equations for the convective fluctuations and assuming some closure hypotheses.

The problem of the interaction between convection and pulsation has been studied by many authors, following different approaches. In this paper, we consider the case where the constrained solution is based on the MLT and the stability of this solution is studied by a linear perturbative method. Two different approaches of the MLT have been proposed, which lead to the same equations at equilibrium but differ when we consider their perturbations. On one hand, the theory of Gough (1965, 1977) is based on Taylor's (1915) and Prandtl's (1925) original analogy between turbulence and the kinetic theory of gases. In this description, the convective elements are accelerated by the buoyancy force over a characteristic length (the Mixing-Length (ML)) and then exchange their thermal energy with the average medium. On the other hand, the theory of Unno (1967) is based on the original ideas of Prandtl (1932). In this description, a turbulent viscosity is introduced. This turbulent viscosity acts in the opposite direction to the buoyancy force, which leads in the stationary case to convective cells with constant velocities. Gabriel et al. (1974, 1975) generalized the theory of Unno (1967) to the case of non-radial modes. Also, some terms neglected by Unno are not neglected in the theory of

⋆ In this paper, we give the final derivations of our time-dependent convection theory. More details about the basic derivations and the implementation in a non-radial non-adiabatic pulsation code are only available as appendices in the electronic version at `http://www.edpsciences.org`

Gabriel (1987, 1996, 1998, 2000). In this paper, we present this theory in a unified form, we detail the basic derivations and approximations and the last improvements. For the first time, we implemented this theory in a non-radial non-adiabatic pulsation code. Preliminary results for the application of our code to δ Sct and γ Dor stars have been presented in Dupret et al. (2004a,b) and Grigahcène et al. (2004). In a subsequent paper in this series (Dupret et al. 2005, hereafter termed Paper II), we will present a more complete and detailed study of the theoretical instability strips for these stars and we will discuss the sensitivity of our results to the convection models.

We note that other time-dependent convection treatments have been proposed. Stellingwerf (1982) derived a nonlocal nonlinear time-dependent treatment of convection, which has been mostly applied to RR Lyrae and Cepheid models. Eggleton (1983) proposed a generalization of Unno's prescription to include composition changes and possible non-local terms. Kuhfuss (1986) proposed a method derived from the hydrodynamic equations by means of the anelastic approximation and diffusion approximation. His method allowed him to describe turbulent convection and derive transport equations for convective mixing. Xiong et al. (1997) developed a non-local theory of convection based on the method of moments. Finally, a completely different approach of the problem of convection-pulsation interaction is based on 3D hydrodynamic simulations (Nordlund & Stein 2001). In this approach, it is no longer required to make an a priori distinction between the convective motions and the oscillations, both are simultaneously and automatically present in the full solution.

In this paper, we present the perturbation of the convection and the last improvements of our theory. In the appendices (only available in the electronic version), the basic derivations leading to these results are presented. The different steps of these derivations are the following. In Appendix A, we give the general hydrodynamic equations. As is generally done in the study of turbulence, we split each physical variable into a mean part and a fluctuating part. Taking the average of the hydrodynamic equations, we obtain the equations for the average medium, as presented in Appendix B. In these equations there appear new correlation terms linked to turbulence: the convective flux $F_{\rm c}$ in the equation of energy conservation, the Reynolds stress tensor $\overline{\rho \boldsymbol{V}\boldsymbol{V}}$ in the equation of momentum conservation, the dissipation rate of kinetic energy of turbulence into heat per gram $\overline{\epsilon_2}$ and the power produced by the buoyancy force $-\overline{\boldsymbol{V}.\boldsymbol{\nabla}p}$, in both the equation of energy conservation and the equation of turbulent kinetic energy conservation. In order to determine these new terms, we take the difference between the general equations and the mean equations, which gives the equations for the convective fluctuations (Appendix C). Following Unno (1967), these equations are then simplified in such a way that, in the stationary case, the usual MLT is recovered. Perturbing the equations for the mean structure gives the equations of linear non-radial non-adiabatic pulsation (Appendix D).

Our perturbed convection theory is presented in the main part of the paper. In Sect. 2, the perturbation of the equations for the convective fluctuations given in Appendix C is presented. We search for local solutions in the form of plane waves and take appropriate averages so that we are able to compute the perturbation of the convective flux (Sect. 2.1), the perturbation of the turbulent pressure (Sect. 2.2) and the perturbation of the turbulent kinetic energy dissipation rate (Sect. 2.3). A well known problem that can arise with the Unno's prescriptions, is the occurrence of short-wavelength spatial oscillations of the thermal eigenfunctions (Baker & Gough 1979; Gonczi & Osaki 1980). We discuss this problem in Sect. 3.1. In Sect. 3.2, we present an important new aspect of our treatment based on a reconsideration of the closure equations perturbation, which enables us for the first time to solve this problem in a local way. Some results are presented for a solar model and two δ Sct models, showing that this new treatment succeeds in solving completely the problem of the short wavelength spatial oscillations. In Sect. 4 we give the contribution of the different terms of our time-dependent convection treatment to the integral expressions for the eigenvalues. Finally, we explain in Appendix E how we have implemented the different time-dependent convection terms of Gabriel's theory in our non-radial non-adiabatic pulsation code.

2. Perturbation of the convection

Stationary solutions of the equations for convective fluctuations (Appendix C, Eqs. (C.2), (C.5) and (C.13)) lead to the classical MLT treatment adopted in our equilibrium models (Gabriel et al. 1974). In order to determine the perturbation of the terms linked to convection we proceed as follows. We perturb Eqs. (C.2), (C.5) and (C.13). Then we search for solutions of the form $\delta\,(\Delta X) = \delta\,(\Delta X)_{\boldsymbol{k}}\ {\rm e}^{{\rm i}\,\boldsymbol{k}\cdot\boldsymbol{r}}\,{\rm e}^{{\rm i}\sigma t}$, assuming constant coefficients. Then we integrate these particular solutions over all values of k_θ and k_ϕ such that $k_\theta^2 + k_\phi^2 = A\,k_r^2$, assuming A constant ($A = 1/2$ for an isotropic turbulence) and that every direction of the horizontal component of $\boldsymbol{k}$ has the same probability. We have to introduce this distribution of $\boldsymbol{k}$ values to obtain an expression for the perturbation of the Reynolds tensor which allows the proper separation of the variables in the equation of motion (Gabriel 1987).

Finally, horizontal averages are computed on a scale larger than the size of the eddies but smaller than the horizontal wavelength of the non-radial oscillations.
After perturbation, Eq. (C.2) becomes, for a given $\boldsymbol{k}$:

$$\boldsymbol{k}\boldsymbol{\cdot}\delta\boldsymbol{V} = 0. \tag{1}$$

The perturbation of Eq. (C.13) gives:

$$\left(\frac{\Delta\rho}{\overline{\rho}} + \frac{\Delta T}{\overline{T}}\right)\frac{{\rm d}\delta\overline{s}}{{\rm d}t} + \frac{{\rm d}\,(\delta\Delta s)}{{\rm d}t} + \delta\boldsymbol{V}\cdot\nabla\overline{s} + \boldsymbol{V}\cdot\delta\,(\nabla\overline{s})$$
$$= -\omega_{\rm R}\,\delta\Delta s - \delta\omega_{\rm R}\Delta s - \delta\left(\frac{\Delta s}{\tau_{\rm c}}\right)\cdot \tag{2}$$

We recall that the term $\Delta s/\tau_{\rm c}$ corresponds to the closure approximation adopted in our MLT treatment for the energy equation (Eq. (C.9)). When $\sigma\,\tau_{\rm c} \ll 1$, convection instantaneously adapts to the changes due to oscillations and we can assume:

$$\delta\left(\frac{\Delta s}{\tau_{\rm c}}\right) = \frac{\Delta s}{\tau_{\rm c}}\left(\frac{\delta\Delta s}{\Delta s} - \frac{\delta\tau_{\rm c}}{\tau_{\rm c}}\right)\cdot \tag{3}$$

This is the treatment adopted in Gabriel (1996). In Sect. 3.2, we will propose another way to perturb $\Delta s/\tau_c$.

In what follows, we use the following notations:

$$B = \frac{i\sigma\tau_c + \Lambda}{\Lambda},$$
$$C = \frac{\omega_R\tau_c + 1}{i\sigma\tau_c + \omega_R\tau_c + 1},$$
$$D = \frac{1}{i\sigma\tau_c + \omega_R\tau_c + 1}\cdot$$

Isolating $(\delta\Delta s)/\Delta s$ in Eq. (2), we obtain:

$$\begin{aligned}\frac{\delta\Delta s}{\Delta s} &= D\Bigg\{-i\sigma\tau_c\frac{Q+1}{Q}\frac{\delta\overline{s}}{c_p} \\ &+(\omega_R\tau_c+1)\left[\frac{\delta V_r}{V_r}+\frac{\boldsymbol{V}\cdot\nabla(\delta\overline{s})}{V_r(\mathrm{d}s/\mathrm{d}r)}-\frac{\boldsymbol{V}\cdot\nabla\xi_r}{V_r}\right] \\ &+\left(\frac{\delta\tau_c}{\tau_c}-\omega_R\tau_c\frac{\delta\omega_R}{\omega_R}\right)\Bigg\}\cdot \end{aligned} \tag{4}$$

Multiplying by V_j/V_r and taking the average gives:

$$\begin{aligned}\frac{\overline{\delta\Delta sV_j}}{\Delta sV_r} = D\frac{\overline{V_rV_j}}{V_r^2}&\left[-i\sigma\tau_c\frac{Q+1}{Q}\frac{\delta\overline{s}}{c_p}+\frac{\delta\tau_c}{\tau_c}-\omega_R\tau_c\frac{\delta\omega_R}{\omega_R}\right] \\ +C&\left[\left(\frac{\nabla_k(\delta\overline{s})}{(\mathrm{d}s/\mathrm{d}r)}-\nabla_k\xi_r\right)\frac{\overline{V_jV_k}}{V_r^2}+\frac{\overline{V_j\delta V_r}}{V_r^2}\right],\end{aligned} \tag{5}$$

where we use the Einstein convention for repeated indices summation.

The perturbation of Eq. (C.5) gives:

$$\begin{aligned}i\sigma\overline{\rho}\delta\boldsymbol{V} &= \delta\left(\frac{\Delta\rho}{\overline{\rho}}\right)\nabla\overline{p}+\frac{\Delta\rho}{\overline{\rho}}\delta(\nabla\overline{p})-\delta(\nabla\Delta p) \\ &-\overline{\rho}\boldsymbol{V}\cdot\delta\nabla\boldsymbol{u}-\frac{\Lambda\overline{\rho}\boldsymbol{V}}{\tau_c}\left(\frac{\delta\overline{\rho}}{\overline{\rho}}-\frac{\delta\tau_c}{\tau_c}\right)-\frac{\Lambda\overline{\rho}\delta\boldsymbol{V}}{\tau_c}\cdot\end{aligned} \tag{6}$$

Taking the divergence of this equation makes the determination of $\delta(\Delta p)$ possible. Substituting the values obtained in Eq. (6) gives, for a given $\boldsymbol{k}$:

$$\begin{aligned}B\frac{\delta V_j}{V_r} &= \frac{A+1}{A}\frac{\delta\left(\frac{\Delta\rho}{\overline{\rho}}\right)}{\frac{\Delta\rho}{\overline{\rho}}}K_{jr} \\ &+\frac{A+1}{A}\left[\left(\frac{\mathrm{d}\overline{p}}{\mathrm{d}r}\right)^{-1}\nabla_i\delta\overline{p}-\nabla_i\xi_r\right]K_{ji} \\ &-\frac{i\sigma\tau_c}{\Lambda}\frac{V_i}{V_r}\nabla_i\xi_lK_{lj}-\left(\frac{\delta\overline{\rho}}{\overline{\rho}}-\frac{\delta\tau_c}{\tau_c}\right)\frac{V_j}{V_r},\end{aligned} \tag{7}$$

where

$$K_{ji} = \delta_{ji}-\frac{k_jk_i}{k_2} \quad \text{and} \quad K_{jr} = \frac{A}{A+1}\frac{V_j}{V_r}\cdot$$

Perturbing Eq. (C.14) (Δp is neglected), we obtain:

$$\delta\left(\frac{\Delta\rho}{\overline{\rho}}\right) = \frac{\Delta\rho}{\overline{\rho}}\left(\frac{\delta\Delta s}{\Delta s}-\frac{\delta Q}{Q}-\frac{\delta c_p}{c_p}\right)\cdot \tag{8}$$

From Eqs. (4), (7) and (8), we find:

$$\begin{aligned}B\frac{\delta V_j}{V_r} &= \frac{V_j}{V_r}\Bigg\{-\frac{\delta Q}{Q}-\frac{\delta c_p}{c_p}-\frac{\delta\rho}{\rho}+\frac{\delta\tau_c}{\tau_c} \\ &+D\Bigg[-i\sigma\tau_c\frac{Q+1}{Q}\frac{\delta s}{c_p}+\frac{\delta\tau_c}{\tau_c}-\omega_R\tau_c\frac{\delta\omega_R}{\omega_R} \\ &+(\omega_R\tau_c+1)\left(\frac{\delta V_r}{V_r}+\frac{V_k}{V_r}\left(\frac{\nabla_k\delta s}{\mathrm{d}s/\mathrm{d}r}-\nabla_k\xi_r\right)\right)\Bigg]\Bigg\} \\ &+\frac{A+1}{A}\left[\left(\frac{\mathrm{d}p}{\mathrm{d}r}\right)^{-1}\nabla_i\delta p-\nabla_i\xi_r\right]K_{ji} \\ &-\frac{i\sigma\tau_c}{\Lambda}\frac{V_i}{V_r}\nabla_i\xi_lK_{lj}.\end{aligned} \tag{9}$$

In the above equations, two terms still have to be determined: $\delta\tau_c/\tau_c$ and $\delta\omega_R/\omega_R$. The perturbation of Eqs. (C.6) and (C.12) gives:

$$\frac{\delta\tau_c}{\tau_c} = \frac{\delta l}{l}-\frac{\overline{\delta V_r}}{V_r}, \tag{10}$$

$$\frac{\delta\omega_R}{\omega_R} = 3\frac{\delta T}{T}-\frac{\delta c_p}{c_p}-\frac{\delta\kappa}{\kappa}-2\frac{\delta\rho}{\rho}-2\frac{\delta l}{l}\cdot \tag{11}$$

On the basis of Eq. (9) it is possible to determine explicitly the different perturbed correlation terms. We recall that the average correlation terms are obtained by integrating the particular solutions over all values of k_θ and k_ϕ such that $k_\theta^2+k_\phi^2 = A\,k_r^2$, and then taking horizontal averages. Considering the case $j = r$, Eq. (9) gives an explicit form for the radial turbulent velocity perturbation:

$$\begin{aligned}\frac{\overline{\delta V_r}}{V_r} &= \frac{1}{B+(i\sigma\tau_c+1)D} \\ &\cdot\Bigg\{-\frac{\delta c_p}{c_p}-\frac{\delta Q}{Q}-\frac{\delta\rho}{\rho}+\frac{\mathrm{d}\delta p}{\mathrm{d}p}-\frac{\mathrm{d}\xi_r}{\mathrm{d}r} \\ &-i\sigma\tau_cD\frac{(Q+1)}{Q}\frac{\delta s}{c_p}+C\left[\frac{\mathrm{d}\delta s}{\mathrm{d}s}-\frac{\mathrm{d}\xi_r}{\mathrm{d}r}\right] \\ &-\frac{A}{A+1}\frac{i\sigma\tau_c}{\Lambda}\left(\frac{\mathrm{d}\xi_r}{\mathrm{d}r}+\frac{1}{A}\frac{\xi_r}{r}-\frac{\ell(\ell+1)}{2A}\frac{\xi_h}{r}\right) \\ &-\omega_R\tau_cD\left(3\frac{\delta T}{T}-\frac{\delta c_p}{c_p}-\frac{\delta\kappa}{\kappa}-2\frac{\delta\rho}{\rho}\right) \\ &+(i\sigma\tau_c+3\omega_R\tau_c+2)D\,\frac{\delta l}{l}\Bigg\}\cdot\end{aligned} \tag{12}$$

Multiplying Eq. (9) (with $j = r$) by V_θ and taking the average gives:

$$\begin{aligned}\frac{\overline{V_\theta\delta V_r}}{V_r^2} &= \frac{\partial Y_\ell^m/\partial\theta}{B-C} \\ &\cdot\Bigg\{\frac{1}{2A}\Bigg[\left(\frac{\delta p}{\mathrm{d}p/\mathrm{d}\ln r}-\frac{\xi_r}{r}+\frac{\xi_h}{r}\right) \\ &+C\left(\frac{\delta s}{\mathrm{d}s/\mathrm{d}\ln r}-\frac{\xi_r}{r}+\frac{\xi_h}{r}\right)\Bigg] \\ &-\frac{1}{2(A+1)}\frac{i\sigma\tau_c}{\Lambda}\left(\frac{\mathrm{d}\xi_h}{\mathrm{d}r}+\frac{\xi_r}{r}-\frac{\xi_h}{r}\right)\Bigg\}\cdot\end{aligned} \tag{13}$$

A similar equation is obtained for $\overline{V_\varphi\,\delta V_r/V_r^2}$. Taking the average of Eq. (9) with $j=\theta$ gives:

$$\begin{aligned}\frac{\overline{\delta V_\theta}}{V_r} = & \frac{\partial Y_\ell^m/\partial\theta}{2\,B}\left[\frac{C}{A}\left(\frac{\delta s}{\mathrm{d}s/\mathrm{dln}\,r}-\frac{\xi_r}{r}+\frac{\xi_h}{r}\right)\right.\\ & +\frac{A+2}{A}\left(\frac{\delta p}{\mathrm{d}p/\mathrm{dln}\,r}-\frac{\xi_r}{r}+\frac{\xi_h}{r}\right)\\ & \left.-\frac{i\sigma\tau_\mathrm{c}/\Lambda}{A+1}\left(\frac{\xi_r}{r}-\frac{\xi_h}{r}+(A+2)\frac{\mathrm{d}\xi_h}{\mathrm{d}r}\right)\right]\\ & +\frac{C}{B}\,\frac{\overline{V_\theta\delta V_r}}{V_r^2}\cdot\end{aligned}\tag{14}$$

More generally, terms of the form $\overline{V_k V_l\,\delta V_j/V_r^3}$ can be obtained by multiplying Eq. (9) by V_kV_l, integrating over $\boldsymbol{k}$ and taking the horizontal average. We do not detail the derivations here. We just notice that, after some algebra, the following useful result can be obtained:

$$\frac{\overline{V_\theta\,\delta V_\theta}}{V_r^2}+\frac{\overline{V_\varphi\,\delta V_\varphi}}{V_r^2}=\frac{1}{A}\frac{\overline{\delta V_r}}{V_r}\cdot\tag{15}$$

2.1. The perturbation of the convective flux

We see in Eqs. (D.5) and (D.8) the appearance of the convective flux perturbation. To obtain it, we perturb Eq. (B.15), which gives:

$$\delta\boldsymbol{F}_\mathrm{c}=\boldsymbol{F}_\mathrm{c}\left(\frac{\delta\rho}{\rho}+\frac{\delta T}{T}\right)+\overline{\rho}\overline{T}\left(\overline{\delta\Delta s\boldsymbol{V}}+\overline{\Delta s\delta\boldsymbol{V}}\right).\tag{16}$$

The radial component of this equation is:

$$\frac{\delta F_{\mathrm{c},r}}{F_{\mathrm{c},r}}=\left(\frac{\delta\rho}{\rho}+\frac{\delta T}{T}\right)+\frac{\overline{\delta\Delta s}}{\Delta s}+\frac{\overline{\delta V_r}}{V_r}\cdot\tag{17}$$

From Eqs. (4), (10) and (11), we obtain an explicit form for the radial component of the perturbation of the convective flux:

$$\begin{aligned}\frac{\delta F_{\mathrm{c},r}}{F_{\mathrm{c},r}} = & \frac{\delta\rho}{\rho}+\frac{\delta T}{T}-i\sigma\tau_\mathrm{c}D\frac{(Q+1)}{Q}\frac{\delta s}{c_\mathrm{p}}+C\left[\frac{\mathrm{d}\delta s}{\mathrm{d}s}-\frac{\mathrm{d}\xi_r}{\mathrm{d}r}\right]\\ & -\omega_\mathrm{R}\tau_\mathrm{c}D\left(3\frac{\delta T}{T}-\frac{\delta c_\mathrm{p}}{c_\mathrm{p}}-\frac{\delta\kappa}{\kappa}-2\frac{\delta\rho}{\rho}\right)\\ & +(i\sigma\tau_\mathrm{c}+2\omega_\mathrm{R}\tau_\mathrm{c}+1)D\frac{\overline{\delta V_r}}{V_r}\\ & +(2\omega_\mathrm{R}\tau_\mathrm{c}+1)D\,\frac{\delta l}{l},\end{aligned}\tag{18}$$

where $\overline{\delta V_r/V_r}$ is given by Eq. (12).
The θ-component of Eq. (16) is:

$$\frac{\delta F_{\mathrm{c},\theta}}{F_{\mathrm{c},r}}=\frac{\overline{\delta\Delta s\,V_\theta}}{\overline{\Delta s\,V_r}}+\frac{\overline{\delta V_\theta}}{V_r}\cdot\tag{19}$$

From Eqs. (5), (13), (14) and using the notation of Eq. (D.7), we find after some algebra:

$$\begin{aligned}\frac{\delta F_{\mathrm{c},h}}{F_{\mathrm{c},r}} = & \frac{C\,(B+1)}{2A\,(B-C)}\frac{\delta s}{\mathrm{d}s/\mathrm{dln}\,r}\\ & +\frac{1}{2AB}\left[\frac{C\,(B+1)}{B-C}+A+2\right]\frac{\delta p}{\mathrm{d}p/\mathrm{dln}\,r}\\ & +\left[\frac{C\,(B+1)\,(2BA+B+1)}{2BA\,(A+1)\,(B-C)}+\frac{B-1}{2B\,(A+1)}+\frac{A+2}{2AB}\right]\\ & \cdot\left(\frac{\xi_h}{r}-\frac{\xi_r}{r}\right)\\ & -\frac{B-1}{2B\,(A+1)}\left[\frac{C\,(B+1)}{B-C}+A+2\right]\frac{\mathrm{d}\xi_h}{\mathrm{d}r}\cdot\end{aligned}\tag{20}$$

2.2. Perturbation of the turbulent pressure

The perturbed turbulent pressure (appearing in Eqs. (D.3), (D.4), (12), (13), (14)) is directly obtained by perturbing Eq. (B.7):

$$\frac{\delta p_\mathrm{t}}{p_\mathrm{t}}=\frac{\delta\rho}{\rho}+2\frac{\overline{\delta V_r}}{V_r},\tag{21}$$

where $\frac{\overline{\delta V_r}}{V_r}$ is given by Eq. (12). Since a term proportional to $\mathrm{d}\delta s/\mathrm{d}s$ is present in Eq. (12) the differential system is one order larger when the perturbation of the turbulent pressure is taken into account in the equation of motion (Eq. (D.3)).

2.3. Perturbation of the rate of dissipation of turbulent kinetic energy into heat

We consider now the perturbation of the last term appearing in the perturbed equation of energy conservation (Eq. (D.5)): $\delta\left(\overline{\epsilon_2+\boldsymbol{V}\cdot\nabla p_\mathrm{th}/\rho}\right)$, where $p_\mathrm{th}=p_\mathrm{g}+p_\mathrm{R}$. This term also appears in the equation of kinetic turbulent energy conservation. We can thus determine it by perturbing Eq. (B.11). Therefore, we get:

$$\delta\left(\overline{\rho\epsilon_2+\boldsymbol{V}\cdot\nabla p_\mathrm{th}}\right)=-i\sigma\rho\,\delta\left(\frac{\overline{\rho\boldsymbol{V}^2}}{2\rho}\right)-i\sigma\overline{\rho\boldsymbol{V}\boldsymbol{V}}\otimes\nabla\boldsymbol{\xi}.\tag{22}$$

The evaluation of the first term gives, using Eq. (15):

$$\begin{aligned}i\sigma\rho\,\delta\left(\frac{\overline{\rho\boldsymbol{V}^2}}{2\rho}\right) & = i\sigma p_\mathrm{t}\left[\frac{\overline{\delta V_r}}{V_r}+\frac{\overline{\delta V_\theta V_\theta}}{V_r^2}+\frac{\overline{\delta V_\varphi V_\varphi}}{V_r^2}\right]\\ & = i\sigma p_\mathrm{t}\frac{A+1}{A}\frac{\overline{\delta V_r}}{V_r}\cdot\end{aligned}\tag{23}$$

The second term of Eq. (22) gives:

$$\overline{\rho\boldsymbol{V}\boldsymbol{V}}\otimes\nabla\boldsymbol{\xi}=p_\mathrm{t}\left[\frac{\mathrm{d}\xi_r}{\mathrm{d}r}+\frac{1}{2A}\left(2\frac{\xi_r}{r}-\ell\,(\ell+1)\,\frac{\xi_h}{r}\right)\right]\cdot\tag{24}$$

And finally we get:

$$\begin{aligned}\delta\left[\overline{\rho\epsilon_2+\boldsymbol{V}\cdot\nabla p_\mathrm{th}}\right]=-i\sigma p_\mathrm{t} & \left[\frac{A+1}{2A}\left(\frac{\delta p_\mathrm{t}}{p_\mathrm{t}}-\frac{\delta\rho}{\rho}\right)\right.\\ & \left.+\frac{\mathrm{d}\xi_r}{\mathrm{d}r}+\frac{1}{2A}\left(2\frac{\xi_r}{r}-\ell\,(\ell+1)\,\frac{\xi_h}{r}\right)\right]\cdot\end{aligned}\tag{25}$$

2.4. Perturbation of the ML

A source of uncertainty in any ML perturbative theory of convection comes from the expression which is adopted for the perturbation of the ML l.

To be coherent with the formula $l = \alpha H_\mathrm{p}$ chosen to compute the static model, it seems normal to assume, when $\sigma\tau_\mathrm{c} \ll 1$, that:

$$\frac{\delta l}{l} = \frac{\delta H_\mathrm{p}}{H_\mathrm{p}} = \frac{\delta p}{p} - \frac{\mathrm{d}\delta p}{\mathrm{d}p} + \frac{\mathrm{d}\xi_r}{\mathrm{d}r}\cdot \tag{26}$$

This formula has been adopted by Schatzman (1956), Kamijo (1962), Unno (1967) and Gough (1977).

On the other hand, it can be expected that the perturbation of the ML becomes negligible when the life-time of the convective elements is much longer than the period of pulsation; this can be reproduced for example by adopting:

$$\frac{\delta l}{l} = \frac{1}{1+(\sigma\tau_\mathrm{c})^2}\frac{\delta H_\mathrm{p}}{H_\mathrm{p}}\cdot \tag{27}$$

In our non-adiabatic pulsation code, Eqs. (26) and (27) can be used optionally. Other ways to perturb the ML have been suggested. Cowling (1935) proposed $\delta l/l = \xi_r/r$. His suggestion was followed by Boury et al. (1964) and Unno (1967). If it is assumed that the Lagrangian coordinates of the starting and arrival points of the convective element remain constant, we get $\delta l/l = \mathrm{d}\xi_r/\mathrm{d}r$. Finally, assuming that the convective element starts with $l = \alpha H_\mathrm{p}$ and then ensures ρl^3 = constant, Unno et al. (1989) proposed:

$$\frac{\delta l}{l} = \frac{1}{1+i\sigma\tau_\mathrm{c}}\left[\frac{\delta H_\mathrm{p}}{H_\mathrm{p}} - \frac{i\sigma\tau_\mathrm{c}}{3}\frac{\delta\rho}{\rho}\right]. \tag{28}$$

We notice that, leaving aside the hypothesis ρl^3 = constant, the real part of Eq. (28) gives Eq. (27).

3. Closure equations and oscillations of the eigenfunctions

3.1. Short wavelength oscillations of the eigenfunctions

A well known problem of this treatment is the occurrence of spatial oscillations of the thermal eigenfunctions with a wavelength much shorter than the ML, which is in contradiction with the basic assumptions of the MLT (Gonczi & Osaki 1980). The same problem also arises in the local ML perturbative theory of Gough (Baker & Gough 1979). These oscillations occur in the part of the convective envelope where $\sigma\,\tau_\mathrm{c} \gg 1$ and most of the energy is transported by convection. The explanation of this phenomenon is the following. Let us consider the conservation of energy equation for a radial mode when most of the energy is transported by convection:

$$i\sigma T\ \delta\overline{s} = -\frac{\mathrm{d}\delta L_\mathrm{c}}{\mathrm{d}m}\cdot \tag{29}$$

Isolating $\mathrm{d}\delta\overline{s}/\mathrm{d}\overline{s}$ in Eq. (18) and considering the case $\sigma\,\tau_\mathrm{c} \gg 1 \gg \omega_\mathrm{R}\tau_\mathrm{c}$, we can write:

$$\frac{\delta L_\mathrm{c}}{L_\mathrm{c}} \simeq \left(\frac{\delta L_\mathrm{c}}{L_\mathrm{c}}\right)_1 + \frac{1}{i\sigma\tau_\mathrm{c}}\frac{\mathrm{d}\delta\overline{s}/\mathrm{d}r}{\mathrm{d}\overline{s}/\mathrm{d}r}\cdot \tag{30}$$

Combining Eqs. (29), (30) and the equilibrium relations of the MLT, we find after some algebra:

$$\frac{\tau_\mathrm{c}}{T}\left[\frac{\mathrm{d}\,(\delta L_\mathrm{c})_1}{\mathrm{d}m} + \frac{2i\pi}{\sigma}\frac{\mathrm{d}(\rho r^2 T\overline{V_r^2})}{\mathrm{d}m}\frac{\mathrm{d}\delta\overline{s}}{\mathrm{d}r}\right] - \frac{1}{2}\frac{l^2}{i\sigma\tau_\mathrm{c}}\frac{\mathrm{d}^2\delta\overline{s}}{\mathrm{d}r^2} + i\sigma\tau_\mathrm{c}\delta\overline{s} = 0. \tag{31}$$

This is the equation of an oscillator whose solutions have a wavelength of: $\sqrt{2}\ l/(\sigma\tau_\mathrm{c})$. In the next section, we propose a new local treatment avoiding this problem.

3.2. A new perturbation of the closure equations

In the method presented above, we have adopted Eq. (3) for the perturbation of the energy closure equation. Many complex physical process, including the whole cascade of energy are extremely simplified in this approach. Therefore, it is clear that much uncertainty is associated to the perturbation of this term. A point to emphasize is that the occurrence of the non-physical spatial oscillations (Sect. 3.1) is directly linked to the perturbation of this closure term. When these oscillations occur ($\sigma\,\tau_\mathrm{c} \gg 1$), the radial derivatives of $\delta\overline{s}$ and $\delta\Delta s$ are of the order of $(\sigma\tau_\mathrm{c}/l)\delta\overline{s}$ and $(\sigma\tau_\mathrm{c}/l)\delta\Delta s$ respectively. Therefore, if we take Eq. (3), we see that the order of magnitude of the perturbation of the right hand side of Eq. (C.9) is $\sigma\tau_\mathrm{c}$ times larger than the left hand side. To have the same order of magnitude, the perturbation of the left hand side should rather be given by:

$$\delta\left(\frac{\Delta s}{\tau_\mathrm{c}}\right) = \beta\,\sigma\,\delta\Delta s - \Delta s\frac{\delta\tau_\mathrm{c}}{\tau_\mathrm{c}^2}, \tag{32}$$

where β is a coefficient of the order of unity. In order to get a formula that switch continuously from Eqs. (3) to (32), we propose to adopt the following expression:

$$\delta\left(\frac{\Delta s}{\tau_\mathrm{c}}\right) = \frac{\Delta s}{\tau_\mathrm{c}}\left((1+\beta\sigma\tau_\mathrm{c})\frac{\delta\Delta s}{\Delta s} - \frac{\delta\tau_\mathrm{c}}{\tau_\mathrm{c}}\right)\cdot \tag{33}$$

With this expression for the perturbation of the closure term, the coefficient $D = (i\sigma\tau_\mathrm{c} + \omega_\mathrm{R}\tau_\mathrm{c} + 1)^{-1}$ in Eq. (4) is replaced by $D = ((i+\beta)\sigma\tau_\mathrm{c} + \omega_\mathrm{R}\tau_\mathrm{c} + 1)^{-1}$. Therefore, in the case $\sigma\,\tau_\mathrm{c} \gg 1$, the coefficient of $\mathrm{d}^2\delta\overline{s}/\mathrm{d}r^2$ in Eq. (31) is approximately $-1/2l^2/((i+\beta)\sigma\tau_\mathrm{c})$ instead of $-1/2l^2/(i\sigma\tau_\mathrm{c})$ and thanks to the real part of β, the non-physical spatial oscillations of the eigenfunctions are no longer present in the solution.

The inclusion of a new free complex parameter β of the order of unity in the perturbation of the closure term of the energy equation is not surprising. As said above, many complex phenomena are neglected in the MLT approach. With this free parameter β, phase lags between the oscillations and the way the turbulence cascade adapts to them are allowed to occur, while they have been neglected in the previous MLT perturbed models.

We illustrate now that our new perturbation of the energy closure equation solves completely the problem of the short wavelength spatial oscillations. In Figs. 1–3, illustration of typical results obtained for a solar model and two δ Sct models at the red border and middle of the instability strip are given.

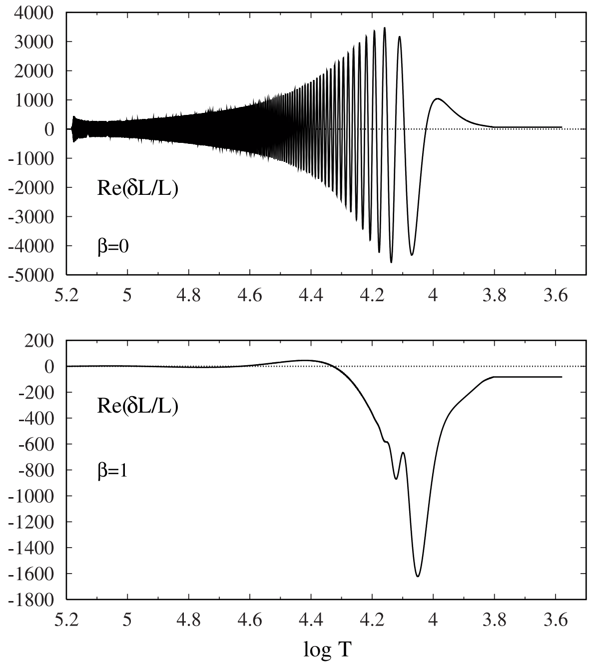

Fig. 1. $\Re(\delta L/L)$ obtained for the $\ell = 2$ p_{20} mode (3024 μHz) of a solar model with our TDC treatment. In the top panel $\beta = 0$ and in the bottom panel $\beta = 1$.

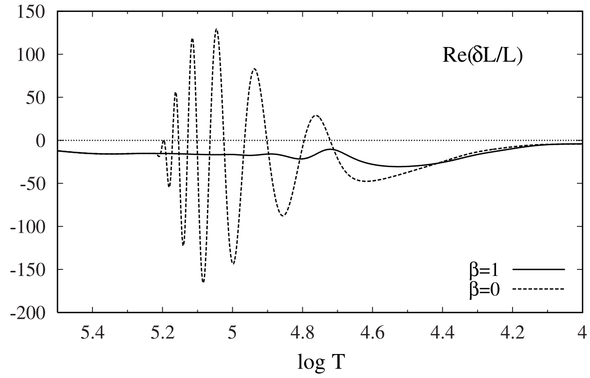

Fig. 2. $\Re(\delta L/L)$ obtained for the radial p_1 mode of a δ Sct model near the red border of the instability strip ($M = 1.8\ M_{\odot}$, $T_{\rm eff} = 6506.5$ K), as obtained with our TDC treatment. Solid line is for $\beta = 1$ and dashed line is for $\beta = 0$.

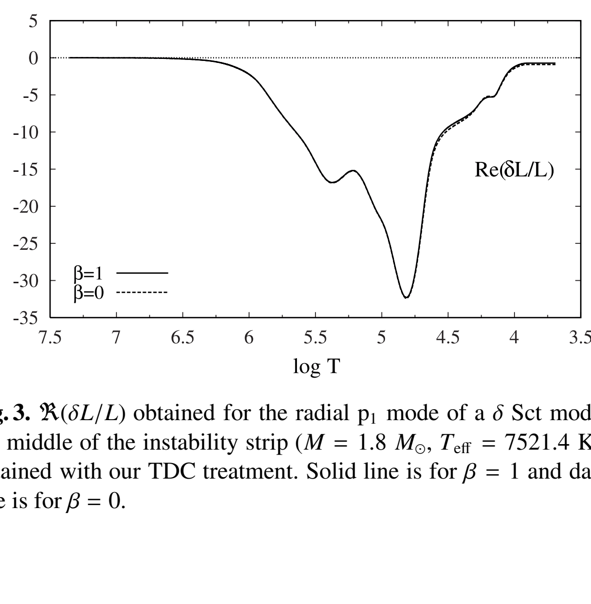

Fig. 3. $\Re(\delta L/L)$ obtained for the radial p_1 mode of a δ Sct model in the middle of the instability strip ($M = 1.8\ M_{\odot}$, $T_{\rm eff} = 7521.4$ K), as obtained with our TDC treatment. Solid line is for $\beta = 1$ and dashed line is for $\beta = 0$.

The solar model was computed using the CESAM evolutionary code (Morel 1997) and the δ Sct models were computed using the evolutionary code CLÉS (Code Liégeois d'Évolution Stellaire). Standard physics is put in these models: the OPAL opacities (Iglesias & Rogers 1996) complemented at low temperatures by the opacities of Alexander & Ferguson (1994), the CEFF equation of state (Christensen-Dalsgaard & Däppen 1992), the MLT treatment of convection (Böhm-Vitense 1958) and the atmosphere models of Kurucz (1998). For the solar model, calibrated values $Y_0 = 0.26766$ and $\alpha = 1.752$ were adopted to fit the solar age, radius and luminosity. The δ Sct models have $M = 1.8\ M_{\odot}$, $Y_0 = 0.28$, $Z = 0.02$, $\alpha = 1.8$ and $\alpha_{\rm ov} = 0.2$. All the terms of our time-dependent convection treatment (perturbation of convective flux, turbulent pressure and dissipation rate of turbulent kinetic energy into heat) were taken into account in our non-adiabatic computations. In these figures, we give the real part of $\delta L/L$ as a function of $\log T$, for the mode $\ell = 0$ p_1 (δ Sct models) and the mode $\ell = 2$ p_{20} (solar model). The computations of our non-adiabatic code are performed from the center to the surface of the star, but for the sake of clarity we only show the results in the relevant region of the convective envelope in Figs. 1 and 2. As usual, these eigenfunctions are normalized in such a way that $\xi_r/r = 1$ at the photosphere. Results are given for two values of the parameter β: 0 and 1. The short-wavelength spatial oscillations are particularly striking for the solar model with $\beta = 0$ (top panel of Fig. 1), simply because $\sigma\,\tau_{\rm c} \gg 1$ in the efficient convective zone of solar-type stars. But our new perturbation of the closure equation succeeds in avoiding these oscillations completely, as shown in the bottom panel of Fig. 1 with $\beta = 1$. For the δ Sct model near the red border of the instability strip (Fig. 2), results with $\beta = 0$ (dashed line) show that the oscillations are also present but to a lesser extent than in the solar case. Again, these oscillations disappear completely with $\beta = 1$ (solid line). However, for most of the stars in the δ Sct instability strip this problem does not occur, simply because $\sigma\,\tau_{\rm c} < 1$ in the convective envelope of typical δ Sct models. As an example, we give in Fig. 3 the results obtained for a model in the middle of the instability strip. This figure illustrates clearly the well known κ-mechanism with the strong decrease of $|\delta L/L|$ in the HeII partial ionization zone. We see that, in this case, the results are very similar for $\beta = 1$ (solid line) and $\beta = 0$ (dashed line). This result is important for the validation of our treatment, showing that when the short wavelength oscillations are not present, the results obtained with our new and old treatments of the closure equation perturbation are very close.

In the second paper of the series (Paper II), we will show that the excitation and damping mechanisms are not very sensitive to this new parameter β for δ Sct and γ Dor stars. For solar-type stars, the confrontation with the observed damping rates of stochastically excited p-modes makes possible to constrain the value of this parameter, as will be discussed in a forthcoming paper.

The solution we have proposed to the problem of the short-wavelength oscillations is local. We note that non-local solutions have also been proposed (Gonczi 1986; Balmforth 1992; Xiong et al. 1997). Non-adiabatic results depend on the specific treatment. More precisely, differences between local and non-local results can occur, but also differences between a given non-local treatment and another non-local treatment, or even between one local treatment and another local treatment. These differences cannot be estimated in a simple way and strongly depend on the type of star considered. The main differences are expected to occur for the thermal eigenfunctions (entropy and flux variations). We showed in Dupret et al. (2004a, Fig. 1) that the locations of the δ Sct theoretical instability strip red edges obtained with our local treatment and with the non-local treatment of Balmforth (1992) and Xiong et al. (1997) are close, although the theories are very different.

4. Integral expressions

It is useful to examine the role played by the different terms of our time-dependent convection treatment in the integral expressions for the frequencies. Multiplying Eq. (D.3) by ξ_r^* ("*" denoting the complex conjugate), using Eq. (D.4) and integrating over the mass of the star, we obtain after some algebra:

$$\begin{aligned}\sigma^2 \int_0^M & \left(|\xi_r|^2 + \ell(\ell+1)|\xi_h|^2\right) \mathrm{d}m \\ = \int_0^M & \left\{ \frac{\delta\rho^*}{\rho}\frac{\delta p}{\rho} + 2\,\Re\left[\xi_r{}^*\frac{\delta\rho}{\rho} g\right] \right. \\ & + \frac{\rho'^*}{\rho}\Phi' - |\xi_r|^2 g \frac{d\ln\rho}{d\,r} \\ & + \frac{2A-1}{A}\frac{p_\mathrm{t}}{\rho}\left[\frac{\xi_r^*}{r}\frac{\mathrm{d}\xi_r}{\mathrm{d}r} + \ell(\ell+1)\frac{\xi_h^*}{r}\left(\frac{\xi_r}{r} - \frac{\xi_h}{r}\right)\right] \\ & \left. + (1/\rho)\left(\xi_r^*\Xi_r + \ell(\ell+1)\xi_h^*\Xi_h\right)\right\}\mathrm{d}m\,. \end{aligned} \tag{34}$$

The imaginary part of this equation gives:

$$\begin{aligned} 2\,\sigma_\mathrm{R}\sigma_\mathrm{I} & \int_0^M \left(|\xi_r|^2 + \ell(\ell+1)|\xi_h|^2\right)\mathrm{d}m \\ = & \int_0^M \mathrm{d}m\,\Im\left\{\frac{\delta\rho^*}{\rho}\frac{\delta p}{\rho} + (1/\rho)\left(\xi_r^*\Xi_r + \ell(\ell+1)\xi_h^*\Xi_h\right)\right. \\ & \left. + \frac{2A-1}{A}\frac{p_\mathrm{t}}{\rho}\left[\frac{\xi_r^*}{r}\frac{\mathrm{d}\xi_r}{\mathrm{d}r} + \ell(\ell+1)\frac{\xi_h^*}{r}\left(\frac{\xi_r}{r} - \frac{\xi_h}{r}\right)\right]\right\}, \end{aligned} \tag{35}$$

where σ_R is the real part of the eigenvalue (angular frequency) and σ_I is the imaginary part (damping rate). The term $-\int_0^M \Im\{\delta\rho^*\delta p/\rho^2\}\mathrm{d}m$ of this equation is the work done by the system during one cycle of pulsation. We consider this term with more attention. We recall that δp is the perturbation of total pressure (including turbulent pressure). To simplify the discussion, we consider the case of a radial mode. From Eq. (D.5), we find:

$$\begin{aligned}\Im\left\{\frac{\delta\rho^*}{\rho}\frac{\delta p}{\rho}\right\} = & \ \Im\left\{\frac{\delta\rho^*}{\rho}\frac{\delta p_\mathrm{t}}{\rho}\right\} + (\Gamma_3 - 1) \\ & \cdot\Re\left\{\frac{\delta\rho^*}{\rho\sigma}\left[\frac{\mathrm{d}\delta L}{\mathrm{d}m} - \delta\epsilon - \delta\left(\overline{\epsilon_2 + \boldsymbol{V}\cdot\frac{\nabla p_\mathrm{th}}{\rho}}\right)\right]\right\}\cdot \end{aligned} \tag{36}$$

From Eq. (25) and assuming isotropic turbulence ($A = 1/2$), we find:

$$\Re\left\{\frac{\delta\rho^*}{\rho\sigma}\,\delta\left(\overline{\epsilon_2 + \boldsymbol{V}\cdot\frac{\nabla p_\mathrm{th}}{\rho}}\right)\right\} = \frac{3}{2}\,\Im\left\{\frac{\delta\rho^*}{\rho}\frac{\delta p_\mathrm{t}}{\rho}\right\}. \tag{37}$$

And finally, Eq. (36) gives:

$$\begin{aligned}\Im\left\{\frac{\delta\rho^*}{\rho}\frac{\delta p}{\rho}\right\} = & \left(1 - \frac{3}{2}(\Gamma_3 - 1)\right)\Im\left\{\frac{\delta\rho^*}{\rho}\frac{\delta p_\mathrm{t}}{\rho}\right\} \\ & + (\Gamma_3 - 1)\,\Re\left\{\frac{\delta\rho^*}{\rho\sigma}\left[\frac{\mathrm{d}\,\delta\,(L_\mathrm{R} + L_\mathrm{c})}{\mathrm{d}m} - \delta\epsilon\right]\right\}\cdot \end{aligned} \tag{38}$$

In agreement with Ledoux & Walraven (1958), we see from this equation that the perturbation of turbulent pressure (in the movement equation) and the perturbation of dissipation rate of turbulent kinetic energy (in the energy equation) have opposite effects on the work integral and thus on the excitation and damping of the modes. In particular, if the gas is completely ionized and radiative pressure is negligible, we have $\Gamma_3 - 1 \simeq 2/3$ and the two terms compensate exactly. Therefore when the perturbation of the turbulent pressure is taken into account, the other terms should also be included and may not be neglected a priori, as their relative importance varies with the behavior of Γ_3 and the assumed shape of the convective eddies linked to the choice of A. As is well known, the regions where $\mathrm{d}\delta L/\mathrm{d}m < 0$ at the hot phase have a driving effect on the oscillations. Many authors neglect the perturbation of convective luminosity in the estimation of this term (frozen convection). However, it is important to take it into account for an accurate analysis of the driving and damping mechanisms in convective zones. We notice also the ambiguity of the frozen convection approximation. It is not clear what would have to be set to zero, the Lagrangian or Eulerian variations of the convective flux, the convective luminosity or the divergence of the convective flux (Pesnell 1990; Li 2000).

5. Conclusion

In this paper, a time-dependent convection treatment is presented, based on the prescriptions of Unno (1967) and the improvements by Gabriel (1974, 1996). A new perturbation of the closure equation that makes possible to avoid the occurrence of unphysical oscillations of the eigenfunctions is proposed, and applications to solar and δ Sct models show that this new treatment works properly. We have numerically implemented in a non-radial non-adiabatic pulsation code the perturbation of the convective flux, turbulent pressure and rate of dissipation of turbulent kinetic energy into heat, according to this theory. The detailed numerical results obtained for δ Sct and γ Dor stars will be presented in a separate paper. Although the perturbation of the full Reynolds stress tensor has not yet been included in our non-adiabatic code, our preliminary results (Dupret et al. 2004a) show that the theory accounts for two important observational constraints: the red edge of the instability strip and a mechanism for driving the g-mode pulsation present in the γ Dor stars.

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Online Material

Appendix A: Hydrodynamic equations

The hydrodynamic equations of mass, momentum, energy conservation and the Poisson equation are respectively:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{A.1}$$

$$\frac{\partial (\rho \boldsymbol{v})}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}\boldsymbol{v}) = -\rho \nabla \Phi - \nabla \cdot \left(\mathsf{P}_{\rm g} + \mathsf{P}_{\rm R}\right), \tag{A.2}$$

$$\frac{\partial (\rho U)}{\partial t} + \nabla \cdot (\rho U \boldsymbol{v}) + \left(\mathsf{P}_{\rm g} + \mathsf{P}_{\rm R}\right) \otimes \nabla \boldsymbol{v} = \rho \epsilon - \nabla \cdot \boldsymbol{F}_{\rm R}, \tag{A.3}$$

$$\nabla^2 \Phi = 4\pi G \rho, \tag{A.4}$$

where ρ is the density, $\boldsymbol{v}$ is the velocity vector, Φ is the gravitational potential, G is the gravity constant, $\mathsf{P}_{\rm g} = p_{\rm g}\mathsf{1} - \beta_{\rm g}$ and $\mathsf{P}_{\rm R} = p_{\rm R}\mathsf{1} - \beta_{\rm R}$ are the gaseous and radiative stress tensors ($\mathsf{1}$ is the identity tensor), $p_{\rm g}$ and $p_{\rm R}$ are the gas and radiative pressures, U is the internal energy, ϵ is the rate of energy generation by nuclear reactions and $\boldsymbol{F}_{\rm R}$ is the radiative flux.

Appendix B: Mean equations

In this section, we follow the same procedure as in Ledoux & Walraven (1958). We split the variables in two parts, describing respectively the average model and the convection. Therefore, we write:

$$y = \overline{y} + \Delta y, \tag{B.1}$$

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{V}, \tag{B.2}$$

where y is any of the variables ρ, p, T, etc. $\overline{y}$ and $\boldsymbol{u}$ are the average values, while Δy and $\boldsymbol{V}$ describe the convection. By convention we put:

$$\frac{\rm d}{{\rm d}t} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla. \tag{B.3}$$

We take horizontal averages of Eqs. (A.1) and (A.2) to get the equations of mass and momentum conservation for the average model. Taking into account that $\overline{\Delta y} = 0$ and $\overline{\rho \boldsymbol{V}} = 0$, we obtain:

$$\frac{\partial \overline{\rho}}{\partial t} + \nabla \cdot (\overline{\rho} \boldsymbol{u}) = 0, \tag{B.4}$$

$$\frac{\partial (\overline{\rho} \boldsymbol{u})}{\partial t} + \nabla \cdot (\overline{\rho} \boldsymbol{u}\boldsymbol{u}) + \nabla \cdot \left(\overline{\rho \boldsymbol{V}\boldsymbol{V}}\right) = -\overline{\rho} \nabla \Phi - \nabla \cdot \left(\overline{\mathsf{P}}_{\rm g} + \overline{\mathsf{P}}_{\rm R}\right). \tag{B.5}$$

We notice the presence of a new tensor $\overline{\rho \boldsymbol{V}\boldsymbol{V}}$, which is the Reynolds stress tensor. We introduce the turbulent viscous tensor by:

$$\overline{\rho \boldsymbol{V}\boldsymbol{V}} = \overline{p}_{\rm t}\,\mathsf{1} - \overline{\beta}_{\rm t}, \tag{B.6}$$

where we choose to define the turbulent pressure by the following equation:

$$\overline{p}_{\rm t} = \overline{\rho \boldsymbol{V}_r^2}. \tag{B.7}$$

At equilibrium, this tensor is diagonal and we define A as:

$$A = \frac{1}{2}\frac{\overline{\rho \boldsymbol{V}_r^2}}{\overline{\rho \boldsymbol{V}_\theta^2}} = \frac{1}{2}\frac{\overline{\rho \boldsymbol{V}_r^2}}{\overline{\rho \boldsymbol{V}_\varphi^2}}. \tag{B.8}$$

In the isotropic case, $A = \frac{1}{2}$.
Neglecting $\beta_{\rm g}$ and $\beta_{\rm R}$, we finally obtain:

$$\overline{\rho}\frac{{\rm d}\boldsymbol{u}}{{\rm d}t} = -\overline{\rho}\, \nabla \Phi - \nabla(\overline{p}_{\rm g} + \overline{p}_{\rm R} + \overline{p}_{\rm t}) + \nabla \cdot \overline{\beta}_{\rm t}. \tag{B.9}$$

Multiplying Eq. (A.2) by $\boldsymbol{v}$, Eq. (B.5) by $\boldsymbol{u}$, taking the difference and then the average, we obtain after some algebra the equation of turbulent kinetic energy conservation:

$$\begin{aligned}\overline{\rho}\frac{\rm d}{{\rm d}t}\left(\frac{1}{2}\frac{\overline{\rho \boldsymbol{V}^2}}{\overline{\rho}}\right) &= -\overline{\left(\beta_{\rm g} + \beta_{\rm R}\right) \otimes \nabla \boldsymbol{V}} \\ &- \overline{\boldsymbol{V} \cdot \nabla \left(p_{\rm g} + p_{\rm R}\right)} - \overline{\rho \boldsymbol{V}\boldsymbol{V}} \otimes \nabla \boldsymbol{u} \\ &- \frac{1}{2}\nabla \cdot \overline{\left(\rho \boldsymbol{V}^2 \boldsymbol{V}\right)} \\ &+ \nabla \cdot \overline{\left(\left(\beta_{\rm g} + \beta_{\rm R}\right) \cdot \boldsymbol{V}\right)}.\end{aligned} \tag{B.10}$$

The last term of this equation is small (Ledoux & Walraven 1958) and the flux of kinetic energy of turbulence given by the term $\frac{1}{2}\nabla \cdot \overline{\left(\rho \boldsymbol{V}^2 \boldsymbol{V}\right)}$ is negligible in the MLT, therefore they are both neglected. We then obtain:

$$\frac{\overline{\rho}}{2}\frac{\rm d}{{\rm d}t}\left(\frac{\overline{\rho \boldsymbol{V}^2}}{\overline{\rho}}\right) = -\overline{\rho \epsilon_2} - \overline{\boldsymbol{V} \cdot \nabla \left(p_{\rm g} + p_{\rm R}\right)} - \overline{\rho \boldsymbol{V}\boldsymbol{V}} \otimes \nabla \boldsymbol{u} \tag{B.11}$$

where $\rho \epsilon_2 = \left(\beta_{\rm g} + \beta_{\rm R}\right) \otimes \nabla \boldsymbol{V}$ is the rate of dissipation of turbulent kinetic energy into heat per unit volume.

The mean equation of energy conservation is obtained by taking the average of Eq. (A.3):

$$\begin{aligned}\overline{\rho}\frac{{\rm d}\overline{U}}{{\rm d}t} + \left(\overline{p}_{\rm g} + \overline{p}_{\rm R}\right) \nabla \cdot \boldsymbol{u} &= -\nabla \cdot \left(\overline{\boldsymbol{F}}_{\rm R} + \overline{\boldsymbol{F}}_{\rm c}\right) + \overline{\rho \epsilon} \\ &+ \overline{\rho \epsilon_2} + \overline{\boldsymbol{V} \cdot \nabla \left(p_{\rm g} + p_{\rm R}\right)}.\end{aligned} \tag{B.12}$$

$\overline{\boldsymbol{F}}_{\rm c}$ is the total flux of energy transported by convection:

$$\overline{\boldsymbol{F}}_{\rm c} = \overline{\left(p_{\rm g} + p_{\rm R} + \rho U\right) \boldsymbol{V}} = \overline{\rho \Delta H \boldsymbol{V}}, \tag{B.13}$$

where H is the enthalpy. Using the entropy s instead of the internal energy, the mean equation of energy conservation finally takes the following form:

$$\overline{\rho}\,\overline{T}\,\frac{{\rm d}\overline{s}}{{\rm d}t} = -\nabla \cdot \left(\overline{\boldsymbol{F}}_{\rm R} + \overline{\boldsymbol{F}}_{\rm c}\right) + \overline{\rho \epsilon} + \overline{\rho \epsilon_2} + \overline{\boldsymbol{V} \cdot \nabla (p_{\rm g} + p_{\rm R})}, \tag{B.14}$$

and neglecting the pressure fluctuations and third order terms, we can write:

$$\overline{\boldsymbol{F}}_{\rm c} = \overline{\rho}\,\overline{T}\,\overline{\Delta s \boldsymbol{V}}. \tag{B.15}$$

We notice that the term $\overline{\rho \epsilon_2 + \boldsymbol{V} \cdot \nabla (p_{\rm g} + p_{\rm R})}$ is present in both Eqs. (B.11) and (B.14) and Eq. (B.11) shows that it is everywhere equal to zero in the equilibrium model. For conciseness, we will call the sum of these two terms the rate of dissipation of turbulent kinetic energy into heat.

Appendix C: Equations for the convective fluctuations

Subtraction of Eqs. (B.4), (B.5) and (B.12) from Eqs. (A.1), (A.2) and (A.3), gives the equations for convection. These are then simplified in such a way that their solution for non-pulsating stars gives the MLT (Böhm-Vitense 1958). This procedure guarantees the compatibility between the theory used to compute our static models and the theory used to evaluate their pulsational stability. As long as static models are computed using MLT, we will also have to use it in stability analysis in order to preserve this consistency.

C.1. Continuity equation

The difference between Eqs. (A.1) and (B.4) gives:

$$\overline{\rho}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Delta\rho}{\overline{\rho}}\right)+\nabla\cdot(\rho\boldsymbol{V})=0. \tag{C.1}$$

In our treatment we use the Boussinesq approximation. In this approximation, the pressure fluctuations are neglected everywhere except in the equation of motion (Eq. (C.5)) and the density fluctuations are neglected in the continuity equation, which gives:

$$\nabla\cdot\boldsymbol{V}=0 \quad \text{or} \quad \nabla\cdot(\rho\boldsymbol{V})=0. \tag{C.2}$$

C.2. Equation of motion

Taking the difference between Eqs. (A.2) and (B.5) and using Eq. (B.4), we find the equation of motion for the convection:

$$\begin{aligned}\overline{\rho}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho\boldsymbol{V}}{\overline{\rho}}\right) &= -\rho\boldsymbol{V}\cdot\nabla\boldsymbol{u}+\frac{\rho}{\overline{\rho}}\nabla\cdot\overline{\left(\mathsf{P}_\mathrm{g}+\mathsf{P}_\mathrm{R}+\rho\boldsymbol{V}\boldsymbol{V}\right)}\\ &\quad -\nabla\cdot\left(\mathsf{P}_\mathrm{g}+\mathsf{P}_\mathrm{R}+\rho\boldsymbol{V}\boldsymbol{V}\right)\\ &= -\rho\boldsymbol{V}\cdot\nabla\boldsymbol{u}+\frac{\Delta\rho}{\overline{\rho}}\nabla\left(\overline{p}_\mathrm{g}+\overline{p}_\mathrm{R}+\overline{p}_\mathrm{t}\right)\\ &\quad -\nabla\Delta\left(p_\mathrm{g}+p_\mathrm{R}+p_\mathrm{t}\right)\\ &\quad -\frac{\Delta\rho}{\overline{\rho}}\nabla\cdot(\overline{\beta}_\mathrm{g}+\overline{\beta}_\mathrm{R}+\overline{\beta}_\mathrm{t})\\ &\quad +\nabla\cdot(\Delta\beta_\mathrm{g}+\Delta\beta_\mathrm{R}+\Delta\beta_\mathrm{t}).\end{aligned} \tag{C.3}$$

We now have to linearize and simplify some terms of this equation to close the problem. This assumption is necessary to recover the MLT but neglects a large number of characteristics of the convection, including the cascade of the energy bound to the coupling of the convective motions at different scales. Following Unno (1967), we assume:

$$\frac{\Delta\rho}{\overline{\rho}}\nabla\cdot(\overline{\beta}_\mathrm{g}+\overline{\beta}_\mathrm{R}+\overline{\beta}_\mathrm{t})-\nabla\cdot(\Delta\beta_\mathrm{g}+\Delta\beta_\mathrm{R}+\Delta\beta_\mathrm{t})=\Lambda\frac{\overline{\rho}\boldsymbol{V}}{\tau_\mathrm{c}}\cdot \tag{C.4}$$

Finally, neglecting $\Delta\rho$ in $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho\boldsymbol{V}}{\overline{\rho}}\right)$, we obtain:

$$\overline{\rho}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}=\frac{\Delta\rho}{\overline{\rho}}\nabla\overline{p}-\nabla\Delta p-\rho\boldsymbol{V}\cdot\nabla\boldsymbol{u}-\Lambda\frac{\overline{\rho}\boldsymbol{V}}{\tau_\mathrm{c}}, \tag{C.5}$$

where $p=p_\mathrm{g}+p_\mathrm{R}+p_\mathrm{t}$. Λ is a dimensionless constant. In our case we take $\Lambda=8/3$ to have compatibility with our equilibrium MLT models. τ_c is the lifetime of the convective elements. It is related to the ML $l=-\alpha(\mathrm{d}\ln p/\mathrm{d}r)^{-1}$ and the radial component of mean turbulent velocity by:

$$\tau_\mathrm{c}=l/\sqrt{\overline{V_r^2}}. \tag{C.6}$$

C.3. Energy equation

Taking the difference between Eqs. (A.3) and (B.12), we obtain the energy equation for the turbulence:

$$\begin{aligned}&\overline{\rho}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho U}{\overline{\rho}}-\overline{U}\right)+\nabla\cdot\left(\rho H\boldsymbol{V}-\overline{\rho H\boldsymbol{V}}\right)\\ &\quad -\boldsymbol{V}\cdot\nabla\left(p_\mathrm{g}+p_\mathrm{R}\right)+\overline{\boldsymbol{V}\cdot\nabla\left(p_\mathrm{g}+p_\mathrm{R}\right)}\\ &\quad +\left(\Delta p_\mathrm{g}+\Delta p_\mathrm{R}\right)\nabla\cdot\boldsymbol{u}-\rho\epsilon_2+\overline{\rho\epsilon_2}\\ &= \rho\epsilon-\overline{\rho\epsilon}-\nabla\cdot\Delta\boldsymbol{F}_\mathrm{R}.\end{aligned} \tag{C.7}$$

We keep only the first order terms in the fluctuations and we work in the Boussinesq approximation. We obtain then for the energy equation:

$$\begin{aligned}\Delta(\rho T)\frac{\mathrm{d}\overline{s}}{\mathrm{d}t}+\overline{\rho T}\frac{\mathrm{d}\Delta s}{\mathrm{d}t}+(\rho T\nabla s)\cdot\boldsymbol{V}-\overline{\rho T\nabla s\cdot\boldsymbol{V}}-\rho\epsilon_2+\overline{\rho\epsilon_2}\\ =\rho\epsilon-\overline{\rho\epsilon}-\nabla\cdot\Delta\boldsymbol{F}_\mathrm{R}.\end{aligned} \tag{C.8}$$

For similar reasons as for the derivation of Eq. (C.4), we assume (Unno 1967):

$$\begin{aligned}\overline{\rho T}\frac{\Delta s}{\tau_\mathrm{c}} &= -\overline{\rho}\overline{T}\boldsymbol{V}\cdot\nabla\overline{s}-\rho\epsilon_2+\overline{\rho\epsilon_2}\\ &\quad +(\rho T\nabla s)\cdot\boldsymbol{V}-\overline{(\rho T\nabla s)\cdot\boldsymbol{V}}.\end{aligned} \tag{C.9}$$

The energy equation becomes:

$$\frac{\Delta(\rho T)}{\overline{\rho T}}\frac{\mathrm{d}\overline{s}}{\mathrm{d}t}+\frac{\mathrm{d}\Delta s}{\mathrm{d}t}+\boldsymbol{V}\cdot\nabla\overline{s}=\frac{(\rho\epsilon-\overline{\rho\epsilon}-\nabla\cdot\Delta\boldsymbol{F}_\mathrm{R})}{\overline{\rho T}}-\frac{\Delta s}{\tau_\mathrm{c}}\cdot \tag{C.10}$$

Since we consider convective envelopes only, we can set $\epsilon=0$. Following the MLT approach, we linearize $\nabla\cdot\Delta\boldsymbol{F}_\mathrm{R}$ as:

$$\nabla\cdot\Delta\boldsymbol{F}_\mathrm{R}=-\omega_\mathrm{R}\Delta s\overline{\rho T}, \tag{C.11}$$

with

$$\omega_\mathrm{R}=\frac{1}{\tau_\mathrm{R}}=\frac{4ac}{3}\frac{\overline{T}^3}{c_\mathrm{p}\overline{\kappa}\overline{\rho}^2\mathcal{L}^2}. \tag{C.12}$$

τ_R is the characteristic cooling time of turbulent eddies due to radiative losses. $\mathcal{L}$ is the characteristic length of the eddies. It is related to the ML l by $\mathcal{L}^2=(2/9)l^2$ to recover the MLT used in our equilibrium stellar models. We finally obtain for the energy equation:

$$\frac{\Delta(\rho T)}{\overline{\rho T}}\frac{\mathrm{d}\overline{s}}{\mathrm{d}t}+\frac{\mathrm{d}\Delta s}{\mathrm{d}t}+\boldsymbol{V}\cdot\nabla\overline{s}=-\frac{\omega_\mathrm{R}\tau_\mathrm{c}+1}{\tau_\mathrm{c}}\Delta s. \tag{C.13}$$

Finally, the equation of state gives (since we neglect Δp):

$$\frac{\Delta\rho}{\overline{\rho}}=\frac{1}{Q}\frac{\Delta s}{c_\mathrm{p}}, \tag{C.14}$$

$$\frac{\Delta T}{\overline{T}}=\frac{\Delta s}{c_\mathrm{p}}, \tag{C.15}$$

where $Q=\left.\frac{\partial\ln T}{\partial\ln\rho}\right|_p$.

Appendix D: Perturbation of the mean structure

In this section, we perturb the equations of the mean structure, which gives the linear non-radial non-adiabatic pulsation equations. As in Eqs. (B.5) and (B.14), coupling terms between convection and pulsation will appear here (perturbation of the convective flux, …). They will be evaluated in Sect. 2. The Lagrangian variation of any quantity y is denoted, for a given spheroidal mode by:

$$\delta y\,(\boldsymbol{r},t) = \delta y\,(r)\exp\left(\mathrm{i}\sigma t\right)Y_\ell^m\left(\theta,\varphi\right),$$

where σ is the angular frequency and Y_ℓ^m the spherical harmonic. In order to be able to distinguish global perturbations from convective motion, we must consider ℓ values small enough so that $r/\ell \gg l$. In what follows, we omit the overlining of the mean quantities when no risk of confusion is present.
The perturbed equation of mass conservation is:

$$\frac{\delta\rho}{\rho} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\xi_r\right) = \ell\left(\ell+1\right)\frac{\xi_h}{r}, \tag{D.1}$$

where we used the notation of Unno et al. (1989) for the displacement vector $\boldsymbol{\xi}$.
The motion equation is obtained by perturbing Eq. (B.9):

$$\begin{aligned}-\sigma^2\rho\,\boldsymbol{\xi} \;&=\; -\delta\rho\nabla\Phi - \nabla\left(\delta p_\mathrm{g} + \delta p_\mathrm{R} + \delta p_\mathrm{t}\right)\\ &\quad +\nabla\boldsymbol{\xi}\cdot\nabla\cdot\beta_\mathrm{t} - \rho\,\nabla\delta\Phi + \delta\left(\nabla\cdot\beta_\mathrm{t}\right).\end{aligned} \tag{D.2}$$

From the definition of A given in Eq. (B.8), we find at equilibrium:

$$\nabla\cdot\beta_\mathrm{t} = -\frac{2A-1}{A}\frac{p_\mathrm{t}}{r}\,\boldsymbol{e}_r.$$

For the perturbation of the divergence of the Reynolds tensor, we use the following notation:

$$\begin{aligned}\delta\left(\nabla\cdot\beta_\mathrm{t}\right) \;&=\; -\,\Xi_r\left(r\right)Y_\ell^m\left(\theta,\varphi\right)\boldsymbol{e}_r\\ &\quad -\,\Xi_h\left(r\right)\left(r\,\boldsymbol{\nabla}_h Y_l^m\left(\theta,\phi\right)\right),\end{aligned}$$

where $\boldsymbol{\nabla}_h$ is the transverse component of the gradient. The radial component of the equation of motion then takes the following form:

$$\begin{aligned}\sigma^2\xi_r \;&=\; \frac{\mathrm{d}\delta\Phi}{\mathrm{d}r} + \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}r}\left(\delta p_\mathrm{g} + \delta p_\mathrm{R} + \delta p_\mathrm{t}\right)\\ &\quad + g\frac{\delta\rho}{\rho} + \frac{2A-1}{A}\frac{p_\mathrm{t}}{r\rho}\frac{\mathrm{d}\xi_r}{\mathrm{d}r} + \frac{\Xi_r}{\rho},\end{aligned} \tag{D.3}$$

where $g = \mathrm{d}\Phi/\mathrm{d}r$ is the gravitational acceleration. The transverse component of the equation of motion is:

$$\begin{aligned}\sigma^2 r\,\xi_h \;&=\; \delta\Phi + \frac{r\,\Xi_h}{\rho} + \frac{\delta p_\mathrm{g} + \delta p_\mathrm{R} + \delta p_\mathrm{t}}{\rho}\\ &\quad + \frac{2A-1}{A}\frac{p_\mathrm{t}}{\rho}\left(\frac{\xi_r}{r} - \frac{\xi_h}{r}\right).\end{aligned} \tag{D.4}$$

From the perturbation of the energy equation (Eq. (B.14)), we find:

$$\begin{aligned}i\sigma T\delta s \;&=\; -\frac{\mathrm{d}\,\delta\left(L_\mathrm{R} + L_\mathrm{c}\right)}{\mathrm{d}m} + \left(\frac{\delta\epsilon}{\epsilon} + \ell\left(\ell+1\right)\frac{\xi_h}{r}\right)\epsilon\\ &\quad + \frac{\ell\left(\ell+1\right)}{4\pi r^3\rho}\left(L_\mathrm{R}\left(\frac{\delta T}{r\left(\mathrm{d}T/\mathrm{d}r\right)} - \frac{\xi_r}{r}\right) - L_\mathrm{c}\frac{\xi_h}{r}\right)\\ &\quad + \frac{\ell\left(\ell+1\right)}{\rho r}\delta F_{\mathrm{c},h} + \delta\left(\overline{\epsilon_2 + \boldsymbol{V}\cdot\frac{\nabla\left(p_\mathrm{g} + p_\mathrm{R}\right)}{\overline{\rho}}}\right).\end{aligned} \tag{D.5}$$

In the diffusion approximation, δL_R is given by:

$$\frac{\delta L_\mathrm{R}}{L_\mathrm{R}} = 2\frac{\xi_r}{r} + 3\frac{\delta T}{T} - \frac{\delta\kappa}{\kappa} - \frac{\delta\rho}{\rho} + \frac{\mathrm{d}\delta T/\mathrm{d}r}{\mathrm{d}T/\mathrm{d}r} - \frac{\mathrm{d}\xi_r}{\mathrm{d}r}\cdot \tag{D.6}$$

In a stellar atmosphere, the diffusion approximation is not valid and we use the treatment proposed in Dupret et al. (2002a) instead of Eqs. (D.5) and (D.6). However, the Eddington approximation or a more general expression for the radiative stress tensor, as used in Gabriel (1996) can also be used. Then Eq. (D.5) must also be slightly modified. For the perturbation of the convective flux we use the following notations:

$$\begin{aligned}\delta\boldsymbol{F}_\mathrm{c} \;&=\; \delta F_{\mathrm{c},r}\left(r\right)\,Y_\ell^m\left(\theta,\phi\right)\,\boldsymbol{e}_r\\ &\quad + \delta F_{\mathrm{c},h}\left(r\right)\,\left(r\nabla_h Y_\ell^m\left(\theta,\phi\right)\right),\end{aligned} \tag{D.7}$$

and

$$\frac{\delta L_\mathrm{c}}{L_\mathrm{c}} = 2\frac{\xi_r}{r} + \frac{\delta F_{\mathrm{c},r}}{F_{\mathrm{c},r}}\cdot \tag{D.8}$$

Appendix E: Implementation in a non-radial non-adiabatic pulsation code

We have implemented the different coupling terms between convection and pulsation in the non-radial non-adiabatic code MAD. This code has already been used for the study of β Cep, SPB, δ Sct and γ Dor stars and includes a detailed treatment of the pulsation in the stellar atmosphere (Dupret et al. 2002b, 2002a). Moya et al. (2004) applied the same treatment to the case of δ Sct stars. The convection-pulsation coupling terms included in the code are the perturbation of the radial and transverse component of the convective flux, the perturbation of the turbulent pressure and finally, the perturbation of the dissipation rate of turbulent kinetic energy into heat. We first recall the numerical structure of the code, and then we detail the implementation of each term.

E.1. Numerical structure of our non-adiabatic code

In our non-adiabatic code, a finite difference method, together with an inverse iteration algorithm is adopted in order to converge on the solution. For all the variables, we use Lagrangian perturbations, except the perturbation of the gravitational potential which is Eulerian. The independent variables of our code are: $\delta L/L$, ξ_r/r, $\delta s/c_v$, $\delta\rho/\rho$ and Φ'. For the discretisation of the equations, we have adopted an interlaced mesh similar to the one proposed initially by Castor (1971), which appears to be very stable from a numerical point of view. The perturbed variables $\delta L/L$ and ξ_r/r are defined on one grid (labelled by 1) while the other variables $\delta s/c_v$, $\delta\rho/\rho$ and Φ' are defined on the other grid (labelled by 2). The two interlaced grids are such that:

$0 < \ldots < r_{1,i-1} < r_{2,i-1} < r_{1,i} < r_{2,i} < r_{1,i+1} < \ldots < R.$

E.2. Discretisation of the convective flux perturbation

A discrete transfer equation relating in a linear way $(\delta L/L)_{1,i}$ to $(\delta s/c_v)_{2,i-1}$, $(\delta s/c_v)_{2,i}$, $(\delta\rho/\rho)_{2,i-1}$, $(\delta\rho/\rho)_{2,i}$, $(\xi_r/r)_{1,i}$ and

$(\xi_r/r)_{1,i+1}$ is required in our code. Decomposing the perturbation of total luminosity into the convective and radiative contributions, we thus need similar relations for $\delta F_{c,r}/F_{c,r}$ and $\delta F_{R,r}/F_{R,r}$. We detail the case of the perturbed convective flux. This term is given by Eqs. (18) and (12). The perturbed equation of state makes possible to relate $\delta T/T$, $\delta c_p/c_p$, $\delta\kappa/\kappa$ and $\delta Q/Q$ to the two independent perturbed variables of our code $\delta s/c_v$ and $\delta\rho/\rho$. The discretisation of the continuity equation (Eq. (D.1)) makes possible to relate $(\xi_h/r)_{2,i}$ to $(\delta\rho/\rho)_{2,i}$, $(\xi_r/r)_{1,i}$ and $(\xi_r/r)_{1,i+1}$. $\delta l/l$ is obtained using Eqs. (26) or (27) optionally. The derivatives of δs and ξ_r are simply estimated by the finite differences between two consecutive points of the grid.

The perturbation of the transverse component of the convective flux is given by Eq. (20). In this equation, the term $(d\xi_h/dr)_{2,i}$ is estimated by taking the finite difference between the continuity equation at two consecutive points, which relate it to $(\delta\rho/\rho)_{2,i}$, $(\delta\rho/\rho)_{2,i+1}$, $(\xi_r/r)_{1,i}$, $(\xi_r/r)_{1,i+1}$ and $(\xi_r/r)_{1,i+2}$. The discrete equation for $(\delta F_{c,h}/F_{c,r})_{2,i}$ is then substituted in the energy equation (Eq. (D.5)).

E.3. Discretisation of the turbulent pressure and kinetic energy dissipation perturbation

The perturbation of the total pressure appears in Eqs. (D.3), (D.4), (12), (20) and (26). As $\delta p = \delta p_g + \delta p_R + \delta p_t$, the evaluation of δp_t is thus required when turbulent pressure is taken into account. Eq. (21) relates it to the perturbed convective velocity, which in turn is given by Eq. (12). However, as $d\delta p/dp$ appears in Eq. (12), we see that the determination of δp_t is implicit. A first solution is to add explicitly in the pulsation code a new variable $(\delta p_t)_{2,i}$ and a new difference equation at each layer. Another solution, rigorous for radial modes only, is to obtain an explicit equation for δp_t by substituting in Eq. (12) and (26) the value of $d\delta p/dp$ given by the equation of motion. We notice that the first solution is more flexible and can be more stable from a numerical point of view.

The discretisation of the turbulent kinetic energy dissipation perturbation is directly deduced from Eq. (25). Then, this equation is substituted in the discrete equation of total energy conservation.