Towards the Red Giants: Renzini's Criterion and Secular Stability

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Abstract: In his attempt to answer the question "Why do stars become red giants ?", Renzini formulated a criterion of thermal stability. After the main sequence phases, this criterion is violated in the envelope and a thermal runaway drives the star in the red giants region of the HR diagram. However, the validity of the criterion has been questioned. In hydrogen shell burning models, the criterion was found inaccurate and it seemed that the very existence of a "simple explanation" was perhaps inappropriate or misleading. The nature of the problem as well as the similarity between Renzini's criterion and an integral form of the secular stability criterion lead us to investigate the link between both criteria. We show that Renzini's criterion is equivalent to a crude approximation of a secular stability criterion, not appropriate to this phase of stellar evolution.

1 Introduction

After the exhaustion of hydrogen in their cores, intermediate mass stars undergo a rapid expansion of their envelope and they become red giants. In an attempt to understand the causes of this expansion, number of authors have proposed simple explanations (Renzini 1984, Yahil and Van den Horn 1985, Applegate 1988, Whitworth 1989, Bhaskar and Nigam 1991, Renzini et al. 1992). The explanation of Renzini and his collaborators rests on a thermal stability criterion that we discuss in this paper.

According to Renzini (1984), the thermal stability of a star depends on the response of the luminosity through any given shell of the envelope to an infinitesimal change of its radius. In a later work (Renzini et al. 1992), this criterion was improved to take into account the feedback effect of the envelope on the nuclear energy production zone. A star is thermally stable if the following inequality holds at each point in the star:

$$\frac{\delta \ln L(r)}{\delta \ln r} - \frac{\delta \ln L_N(r)}{\delta \ln r} > 0. \tag{1}$$

$\delta L(r)$ and $\delta L_N(r)$ are respectiveley the variation of the luminosity and the variation of the contribution of nuclear reactions to the luminosity which would result from an infinitesimal expansion δr of the layer located at point r. The criterion is fulfilled during the main sequence phase of intermediate mass stars. But as the evolution proceeds, during hydrogen shell burning, the criterion is violated in the envelope. This instability, primarily controlled by the behaviour of the opacity, marks the onset of a rapid expansion of the envelope which drives the star in the red giants region of the Hertzsprung-Russell diagram. This thermal runaway would eject the envelope if it were not quenched later by the development of convection.

However this point of view has been criticized on the basis of numerical computations. Weiss (1989) has computed standard stellar evolutions and has performed numerical experiments. The relation between Renzini's criterion and the envelope behaviour was found ambiguous. The agreement is better after the maximum luminosity is reached. Weiss concludes that Renzini's criterion merely reflects the changes in luminosity and radius that take place during the evolution but does not demonstrate any property of the star. Iben (1993) performed several numerical experiments in an attempt to isolate those factors which do or do not play a role in the expansion of the envelope. According to him, the expansion of the envelope results from a complicated interplay between the core, the envelope and the hydrogen burning shell ; a "simple explanation" of the phenomenon can be an inappropriate and misleading description.

2 Secular stability

Let us state the problem on the firm basis of the well established linear theory of stellar stability. If we artificially break the thermal equilibrium of a stellar model while maintaining hydrostatic equilibrium, the model will subsequently evolve on a Kelvin-Helmholtz time scale. Whether the model evolves towards or away from its initial equilibrium configuration, this configuration is said to be *secularly stable* or *secularly unstable.* The time evolution of a linear secular mode can be described by a factor of the form e^{st} and the coefficient s can be written (Ledoux 1963 and 1969):

$$s = -\frac{\int_0^M (\Gamma_3 - 1)\frac{\overline{\delta\rho}}{\rho}(\delta\epsilon - \frac{d\,\delta L(m)}{dm})\,dm}{\int_0^M \left\{ c^2 r^2 \left| \frac{d}{dr}\left(\frac{\delta r}{r}\right)\right|^2 - \frac{r}{\rho}\frac{d}{dr}\left[(3\Gamma_1 - 4)P\right]\left|\frac{\delta r}{r}\right|^2 \right\} dm}. \tag{2}$$

As the denominator is positive, the considered secular mode is stable if

$$\Re \int_0^M (\Gamma_3 - 1)\frac{\overline{\delta\rho}}{\rho}(\delta\epsilon - \frac{d\,\delta L(m)}{dm})\,dm > 0. \tag{3}$$

A model is secularly stable if all its secular modes satisfy this condition. One could be tempted to state the condition for secular stability in the following terms: a stellar model is secularly stable if criterion (3) is fulfilled for all linear perturbations compatible with the continuity equation, the hydrostatic equilibrium equation and the transfer equation. However, such a statement is false because the integral operator defining the quadratic form (3) is not self-adjoint (it is easy to find counterexamples for simpler problems).

When the computation of secular stability modes was out of reach of the available computing facilities, condition (3) was generally tested for only one secular mode, which was supposed to be not too far from a homologous transformation:

$$\frac{\delta r}{r} = \alpha\,, \quad \frac{\delta\rho}{\rho} = -3\alpha\,, \quad \frac{\delta P}{P} = -4\alpha\,, \quad \ldots \quad \text{where } \alpha = \text{const.} \tag{4}$$

Substituting into criterion (3) and neglecting the variation of Γ_3 throughout the star, one obtains a crude secular stability criterion: in a homologous perturbation,

$$\int_0^M \delta\epsilon\,dm - \delta L \quad \text{and} \quad \frac{\delta r}{r} \quad \text{have opposite signs.} \tag{5}$$

This means that in a homologous contraction, the total variation of the nuclear energy production must exceed the variation of the luminosity.

One can further simplify the criterion if the star is entirely radiative and if the opacity and energy generation laws assume the simple forms

$$\kappa = \kappa_0 \rho^m T^n \tag{6}$$
$$\epsilon = \epsilon_0 \rho^\mu T^\nu . \tag{7}$$

The criterion reduces to the well-known and extensively used Jeans' secular stability criterion (Jeans, 1928):

$$3\mu + \nu + 3m + n > 0. \tag{8}$$

In the course of their evolution, stars go through phases where their structure departs markedly from thermal equilibrium. During these phases, the conversion of gravitational energy into thermal energy and the reverse process become significant when compared with the nuclear energy generation. This situation can be formally coped with by adding a gravitational energy term $\epsilon_g = -T\, dS/dt$ to the nuclear term in the equation of thermal equilibrium. Such models are called quasi-static models.

The concept of secular stability has be extended to quasi-static models (Gabriel 1972, Noels 1972). These authors have also shown that the results of the secular stability analysis must be understood as linear approximations to the evolution.

We have computed a number of standard stellar evolutionary tracks for a few masses between 2 and 15 $M_\odot$ and for a solar chemical composition (X=0.71673, Y=0.26571 and Z=0.01756) up to the red giant phase (Figure 1). We did not carry out a linear secular stability analysis. The stability of the 2 $M_\odot$ and 15 $M_\odot$ models was however investigated in the following way. At each point of their evolutionary track, we let them evolve at constant chemical composition, without turning off the nuclear energy source. If the model evolves towards a nearby configuration, we say that it is stable, otherwise we consider it as unstable. This definition is a bit loose, but in practice, it is easy to handle. We guess that this sort of stability must be close to secular stability but we did not investigate the link between them. In Figure 1, the arrows labelled QS indicate the point where the models become quasi-static and the onset of the instability is indicated by the arrows labelled I. Figure 1 shows clearly that the beginning of the crossing of the HR diagram (i.e. the end of the main sequence phase), the departure from thermal equilibrium and the onset of instability occur independently. It is therefore difficult to consider this instability as the primordial cause of the evolution of a star towards the red giant phase.

3 Renzini's criterion as an approximate secular stability criterion

For the following discussion, we rewrite Renzini's criterion in a different form. The left-hand side member of criterion (1) may be written

$$-\frac{\gamma \int_0^m \delta\epsilon\, dm - \delta L(m)}{L(m)\dfrac{\delta r}{r}} \quad \text{with} \quad \gamma = \frac{L(m)}{L_N(m)}. \tag{9}$$

Criterion (1) is thus equivalent to require that

$$\gamma \int_0^m \delta\epsilon\, dm - \delta L(m) \quad \text{and} \quad \frac{\delta r}{r} \quad \text{have opposite signs} \tag{10}$$

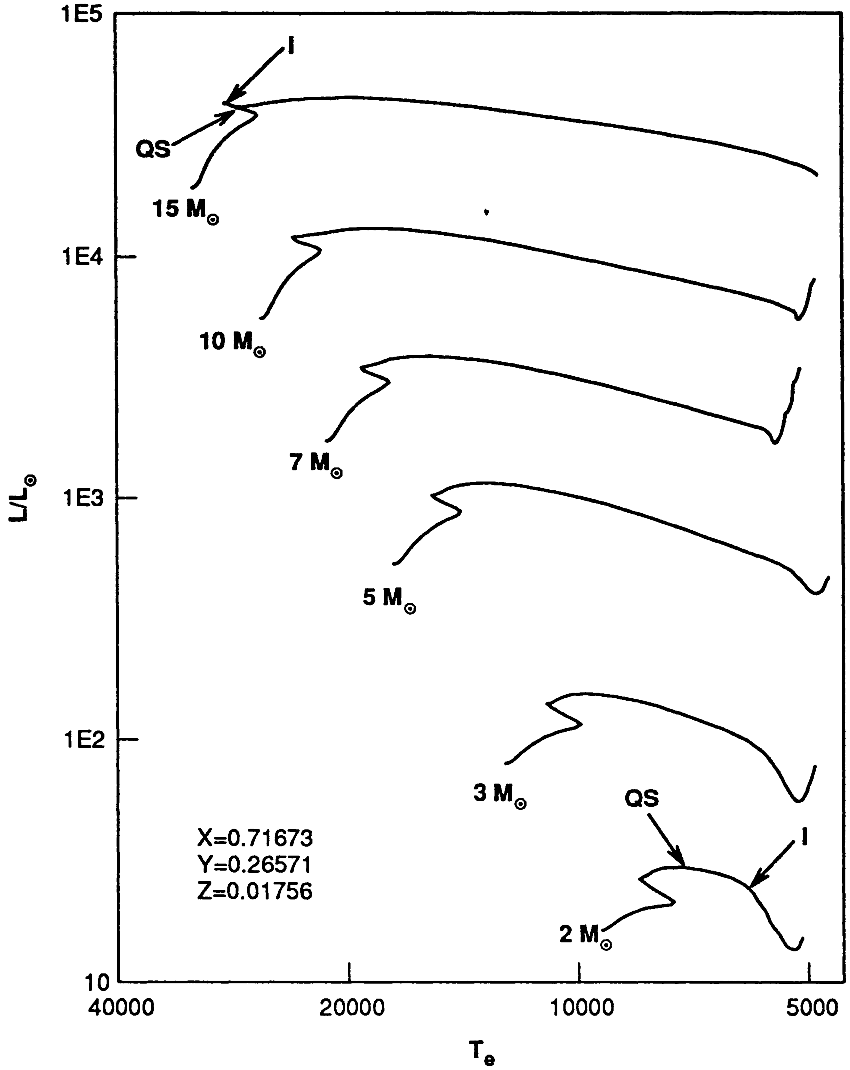

Figure 1: Evolutionary tracks from main sequence to red giants. At point QS the models become quasi-static and at point I they become unstable.

at each point in the star. For a star in thermal equilibrium $\gamma = 1$ and γ departs significantly from 1 only when the star is markedly out of thermal equilibrium. As Renzini's criterion does not include a $\delta\epsilon_g$ term, we restrict the following discussion to models in thermal equilibrium. In this case, Renzini's criterion reduces to

$$\int_0^m \delta\epsilon \, dm - \delta L(m) \quad \text{and} \quad \frac{\delta r}{r} \quad \text{have opposite signs} \tag{11}$$

at each point in the star.

The approximate secular stability criterion (5) is thus criterion (11) applied to the surface point of the model. On the other hand, criterion (11) applied at point r gives the contribution of the sphere of radius r to the secular stability (or instability) of the model according to criterion (5).

The link between Renzini's criterion (11) and secular stability can be made tighter in the following way. Instead of a homologous perturbation affecting the whole star, let's consider a whole family (depending on the parameter r^*) of perturbations described by

$$\begin{cases} \dfrac{\delta r}{r} = \alpha & \text{and} \quad \dfrac{\delta \rho}{\rho} = -3\alpha \quad \text{if} \quad r \leq r^* \\ \dfrac{\delta r}{r} = \alpha \left(\dfrac{r^*}{r}\right)^3 & \text{and} \quad \dfrac{\delta \rho}{\rho} = 0 \quad \text{if} \quad r > r^* \end{cases} \tag{12}$$

Ignoring the variations of Γ_3, criterion (3) of secular stability becomes

$$\int_0^{m^*} \delta\epsilon \, dm - \delta L(m^*) \quad \text{and} \quad \left(\frac{\delta r}{r}\right)^* \quad \text{have opposite signs} \tag{13}$$

at each point r^* in the star.

This criterion is exactly the same as Renzini's criterion (11). The interpretation in terms of secular stability brings a profound weakness of Renzini's criterion to light. As the eigenfunctions of the secular stability problem mimic the evolution (Gabriel 1972, Noels 1972), a good approximation to an eigenfunction in this phase of evolution should allow the contraction of the core and the expansion of the envelope. This is not the case for the functions of the special family used to write down criterion (13). It may be argued that these functions form a basis for all admissible secular perturbations. However the argument must be rejected: as the problem of secular stability is not self-adjoint, the exact criterion (3) and Renzini's criterion (11) might lead to opposite conclusions.

4 Conclusion

Renzini's criterion is a crude approximate secular stability criterion. From this point of view it suffers from two flaws: it does not take thermal imbalance into account (no $\delta\epsilon_g$ term) and it is too crude an approximation to be useful.

On the basis of our computations and computations carried out by other authors, we believe that the departure from thermal equilibrium or the onset of secular instability do not explain in any way why stars become red giants. These instabilities may, possibly, explain some features of the evolution towards the red giants but the primordial cause of the envelope expansion is still unknown.

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